BRDF models and their use in Global Illumination Algorithms

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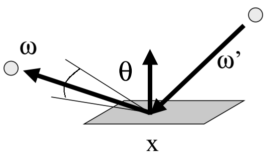
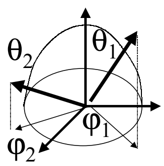
Definition of the BRDF



$$\frac{w(\omega', x, \omega)}{\cos\theta} = f_r(\omega', x, \omega)$$

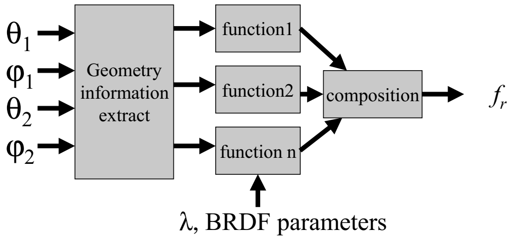
Bidirectional Reflectance Distribution Function
BRDF: $f_r(\omega', x, \omega)$ [1/sr]

Representation of Measured BRDF data



BRDF is a 5-variate function:
$\theta_1, \varphi_1, \theta_2, \varphi_2, \lambda$
Table size: 100x100x100x100x10 = 10^9

Mathematical BRDF models



Properties of BRDFs

- 1. Positive: probability times cosine angle
- 2. Symmetric (reciprocal) - Helmholtz

$$f_r(\omega', x, \omega) = f_r(\omega, x, \omega')$$

- 3. Energy conserving:
  - the reflected energy is less than the incoming energy
  - the incoming photon is reflected with a probability less than 1

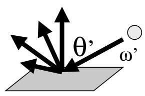
Albedo

- Probability that a photon coming from ω' is reflected to any direction (not absorbed):

$$a(x,\omega') = \int w(\omega',x,\omega)\, d\omega = \int f_r(\omega',x,\omega) \cos\theta \, d\omega$$

- Energy conservation:

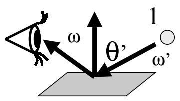
$$a(x, \omega') < 1$$



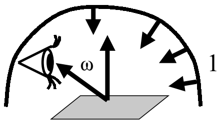
Visibility of the albedo

- Reflection of homogenous illumination

$$a(x,\omega) = \mathcal{T}\, 1 = \int f_r\,(\omega', x, \omega)\cos\theta' d\omega'$$



$L^{ref} = 1\, f_r\,(\omega', x, \omega)\cos\theta'$



$L^{ref} = 1\; a(x,\omega)$

Physically plausible BRDFs

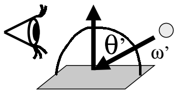
- Positive, symmetric, energy conserving
- do not violate physics
- make the transport operator a contraction
  - Proof with the infinite norm: $||f|| = \max |f|$

$$||\mathcal{T}L|| = \max \int L(h(x,-\omega),\omega) f_r \cos\theta' d\omega' \le$$

$$\max \int f_r \cos\theta' d\omega' \bullet \max L = \max a(x,\omega) \bullet ||L||$$

Diffuse reflection

- Radiance is independent of the out direction

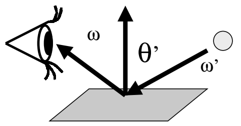


- Helmholtz: independent of the in direction
- BRDF is constant:

$$f_r\,(\omega', x, \omega) = k_d(\lambda)$$

Lambert's law

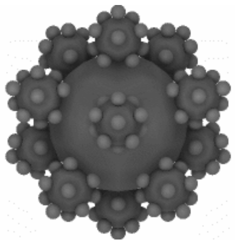
- Response to a point-lightsource

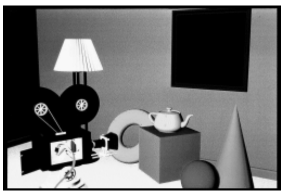


$$L^{ref} = L^{ir} \; k_d \cos\theta'$$

```
class Diffuse {
   Color Kd;
public:
   Color BRDF(Vec& L, Vec& N, Vec& V) { return Kd; }
   Color Albedo( Vec& N, Vec& V ) { return Kd*M_PI; }
};
```

Image of diffuse objects





Physical plausibility of the diffuse BRDF

$$f_r(\omega',x,\omega) = k_d$$

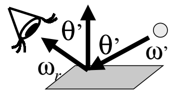
- Positive: ✓
- Symmetric: ✓
- Energy conserving: $k_d < 1/\pi$

$$a(x,\omega') = \int k_d \cos\theta \, d\omega = \int_\phi \int_\theta k_d \cos\theta \underbrace{\sin\theta \, d\theta d\phi}_{d\omega} =$$

$$k_d \, 2\pi \int_\theta \cos\theta \sin\theta \, d\theta = k_d \, \pi$$

BRDF of ideal reflection

- Radiance is reflected only to the ideal mirror direction



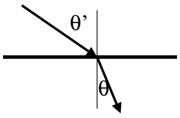
$$L^{ref} = L^{ir}\ k_r$$

- BRDF is a Dirac-delta function:

$$f_r\,(\omega',x,\omega) = \delta(\omega-\omega_r)\,k_r/\cos\theta'$$

Reflection of ideal mirrors: k_r - the Fresnel function

$$F_{\|} = \left|\frac{\cos\theta - (n+kj)\cos\theta'}{\cos\theta + (n+kj)\cos\theta'}\right|^2 \qquad F_{\perp} = \left|\frac{\cos\theta' - (n+kj)\cos\theta}{\cos\theta' + (n+kj)\cos\theta}\right|^2$$

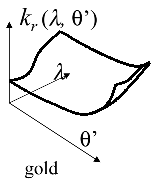


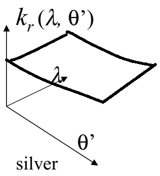
$$n = \frac{\sin\theta'}{\sin\theta}$$

Snellius-Descartes law of refraction
n = Relative speed of the wave

If the light is not polarized

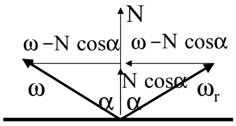
$$k_r(\lambda, \theta') = \frac{\left|F_{\|}^{1/2}\,\mathbf{E}_{\|} + F_{\perp}^{1/2}\,\mathbf{E}_{\perp}\right|^2}{\left|\mathbf{E}_{\|} + \mathbf{E}_{\perp}\right|^2} = \frac{F_{\|} + F_{\perp}}{2}$$





Calculation of reflection direction

Law of reflection:
angle of the outgoing light equals to the angle of the incoming light and the incoming beam, outgoing beam and the surface normal are in a single plane.



$$\omega_r = 2 \cos \alpha\, N - \omega$$

$L = \omega_r,\ V = \omega$

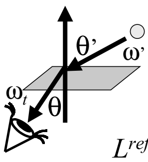
```
class Reflector {
   Color  Kr;
public:
   BOOL ReflectDir(Vec& L, N, V) {
      L = N * (N * V) * 2 - V;
      return TRUE;
   }
   Color Albedo( Vec& N, Vec& V ) {
      return Kr;
   }
};
```

BRDF of ideal refraction

- Radiance is refracted only to the ideal refraction direction

$$n = \frac{\sin\theta'}{\sin\theta}$$

Snellius-Descartes law of refraction

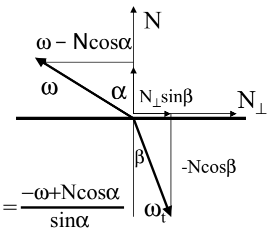


$$L^{refract} = L^{ir}\ k_t$$

- BRDF is a Dirac-delta function:

$$f_r(\omega', x, \omega) = \delta(\omega - \omega_t)\, k_t / \cos\theta'$$

Calculation of refraction direction



$$N_\perp = \frac{-\omega + N\cos\alpha}{\sin\alpha}$$

$$n = \frac{\sin \alpha}{\sin \beta}$$

Snellius-Descartes law of refraction

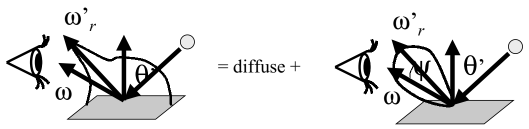
$$\omega_t = N\left(\cos\alpha/n - \sqrt{1 - (1 - \cos^2\alpha)/n^2}\right) - \omega/n$$

Refraction class

L = ω_t, V= ω

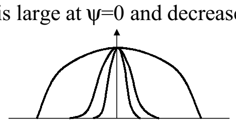
```
class Refractor {
   Color  Kt;
   double n;
public:
   BOOL RefractionDir(Vec& L, Vec& N, Vec& V, BOOL out) {
      double cn = n;
      if ( !out ) cn = 1.0/cn;
      double cosa = N * V;      // Snellius-Descartes law
      double disc = 1 - (1 - cosa * cosa) / cn / cn;
      if (disc < 0) return FALSE;
      L = N * (cosa / cn - sqrt(disc)) - V / cn;
      return TRUE;
   }
};
```

BRDF of specular reflection: Phong model



A function is needed that is large at ψ=0 and decreases rapidly

$k_s \cos^n \psi$

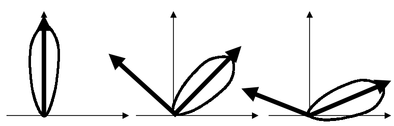


Original Phong model

$L^{ref} = L^{ir} k_s \cos^n \psi$

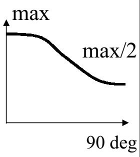
$f_r(\omega',x,\omega) = k_s \cos^n \psi / \cos\theta'$

- Not symmetric!

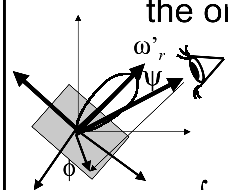


Lobes of reflected radiance

”Albedo”: Probability of reflection



Computation of the "albedo" of the original Phong



$d\omega = \sin\psi \, d\psi \, d\phi$

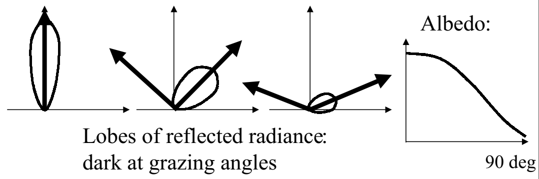
Over estimate:
directions that go to the object !
In the reflection lobe: $\theta' \approx \theta$

$$a(x,\omega') = \int k_s \cos^n\psi / \cos\theta' \cdot \cos\theta \, d\omega \approx \int_\phi \int_\psi k_s \cos^n\psi \underbrace{\sin\psi \, d\psi \, d\phi}_{d\omega}$$

$$a(x,\omega') \approx 2\pi k_s/(n+1) \cdot \text{Cutting}(1..0.5)$$

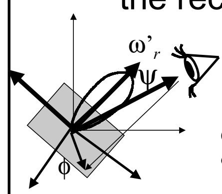
Reciprocal Phong model

$f_r(\omega', x, \omega) = k_s \cos^n \psi$
$L^{ref} = L^{ir} k_s \cos^n \psi \cos\theta'$



Lobes of reflected radiance:
dark at grazing angles

Computation of the albedo of the reciprocal Phong

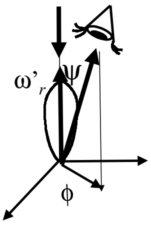


$d\omega = \sin\psi \, d\psi \, d\phi$

Over estimate:
directions that go to the object !

$$a(x,\omega') = \int k_s \cos^n\psi \cos\theta \, d\omega = \int_\phi \int_\psi k_s \cos^n\psi \sin\psi \cos\theta \, d\psi \, d\phi$$

Albedo of the reciprocal Phong: perpendicular illumination



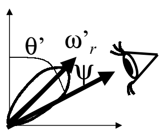
$\psi = \theta$!

$a(x,\omega') = \int k_s \cos^n\psi \cos\theta \, d\omega =$

$\int_\phi \int_\psi k_s \cos^{n+1}\psi \sin\psi \, d\psi \, d\phi =$

$a_{max}(x,\omega') = 2\pi k_s / (n+2)$

Albedo of the reciprocal Phong: arbitrary illumination, shiny surfaces



θ is approximately constant in the reflection lobe and equals to θ'

$a(x,\omega') = \int k_s \cos^n\psi \cos\theta \, d\omega =$

$\cos\theta' \int k_s \cos^n\psi \, d\omega =$

$a(x,\omega') \approx 2\pi k_s / (n+1) \cos\theta' \cdot \text{Cutting}(1..0.5)$

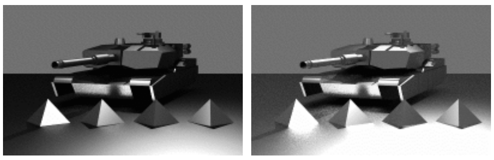
Reciprocal Phong BRDF class

```
class Phong {
   Color  Ks;
   double shine;
public:
   Color BRDF(Vec& L, Vec& N, Vec& V) {
      double cos_in = L * N;
      if (cos_in > 0 && ks() != 0) {
         Vec R = N * (2.0 * cos_in) - L;
         double cos_refl_out = R * V;
         if (cos_refl_out > 0)  return (Ks * pow(cos_refl_out, shine));
      }
      return SColor(0);
   }
   Color Albedo(Vec& N, Vec& V)
         { return ks()*2*M_PI/(shine+2) * (N*V); }
};
```

Pumping up the reciprocal Phong

- Metals: albedo does not decline at grazing angles
- $f_r(\omega',x,\omega) = k_s \cos^n \psi / X$
- X is like cosθ' but symmetric
  - X = (cosθ'+ cosθ)/2: reflection is 2 times greater at grazing angles than at perpendicular illumination
  - X = cosθ'·cosθ: albedo ∞ at grazing angles
  - $X = \sqrt{\cos\theta' \cdot \cos\theta}$: albedo ∞ at grazing angles

Reciprocal and the pumped-up models

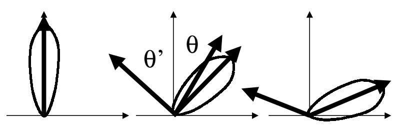


Max Phong model

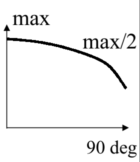
$$f_r(\omega',x,\omega) = k_s \cos^n \psi / \max(\cos\theta', \cos\theta)$$

$$L^{ref} = L^{ir}\, k_s \cos^n\psi \qquad \text{if } \theta' < \theta$$

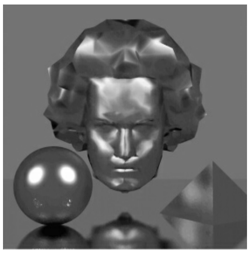
$$L^{ref} = L^{ir}\, k_s \cos^n\psi \cos\theta'/\cos\theta \qquad \text{if } \theta' > \theta$$

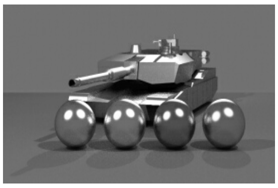


Lobes of reflected radiance



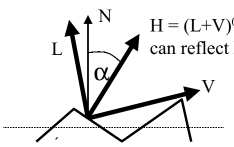
Modified Phong + Fresnel Rendering: path tracing





Cook-Torrance model

- Physically based model: surface is a collection of randomly oriented perfect mirrors of the same size *f*

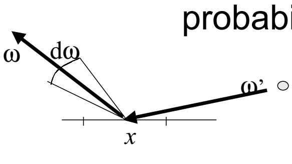


$H = (L+V)^0$ normal vector of microfacets that can reflect L to V

Beckmann distribution:

$$\Pr(H) = \frac{\exp(-(\tan^2 \alpha)/m^2)}{m^2 \cos^2 \alpha}$$

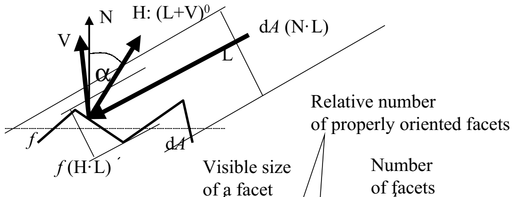
Cook Torrance reflection probability



$w(\omega', x, \omega)\, d\omega =$ Pr{photon goes to $\omega \pm d\omega$ | comes from ω'}

- Reflection is an AND of the following events:
  - microfacet met by the photon is properly oriented
  - no masking and shadowing takes place
  - photon is not absorbed

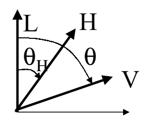
Probability of proper orientation



$$\text{Pr(orientation)} = \frac{f\cdot(H\cdot L)\ \text{Pr}(H)\, d\omega_H\ dA/f}{dA\,(N\cdot L)}$$

Total visible area

$d\omega_H/d\omega$



$$d\omega_H = \sin\theta_H\, d\theta_H d\phi_H$$

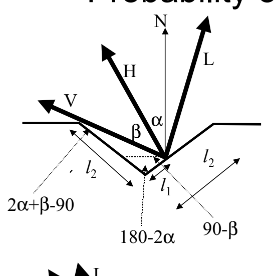
$$d\omega = \sin\theta\, d\theta d\phi$$

$\phi = \phi_H$, $\theta = 2\theta_H$

$$d\omega_H/d\omega = \sin\theta_H / 2\sin 2\theta_H = 1/4 \cos\theta_H = 1/(4(H\cdot L))$$

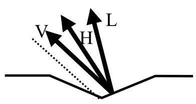
$$\text{Pr(orientation)} = \frac{\text{Pr}(H)}{4\,(N\cdot L)}\, d\omega$$

Probability of masking



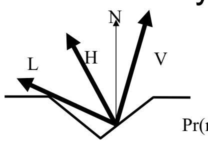
$$\text{Pr(not masking)} = 1 - l_1 / l_2 = 1 - \sin(2\alpha+\beta-90)/\sin(90-\beta) = 1+\cos(2\alpha+\beta)/\cos(\beta) =$$

$$\frac{\cos(\beta)+\cos(2\alpha+\beta)}{\cos(\beta)} = \frac{2\cos(\alpha)\cos(\alpha+\beta)}{\cos(\beta)}$$



$$\text{Pr(not masking)} = \min\left(2\frac{(N\cdot H)\,(N\cdot V)}{(V\cdot H)}, 1\right)$$

Probability of shadowing



Symmetry: mask ⇔ shadow
L ⇔ V

$$\Pr(\text{not shadow}) = \min\left(2\frac{(N\cdot H)(N\cdot L)}{(L\cdot H)}, 1\right)$$

Pr(no shadow AND no mask) = G(N,L,V)=

$$\min\left(2\frac{(N\cdot H)(N\cdot V)}{(V\cdot H)}, 2\frac{(N\cdot H)(N\cdot L)}{(L\cdot H)}, 1\right)$$

Cheat!!!:
albedo is infinite
at grazing angles

Cook-Torrance BRDF

$\Pr(\text{not absorb}) = F((L\cdot H), \lambda)$ Fresnel function

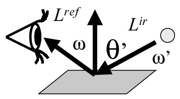
$f_r(L,V)$=Probability of reflection/$d\omega$/$(N\cdot L)$

$$f_r(L,V) = \frac{\Pr(H)}{4(N\cdot V)(N\cdot L)} \cdot G(N, L, V) \cdot F((L\cdot H), \lambda)$$

Physically based versus Empirical models

- Physically based:
  - structural validity
  - difficult to compute:
    - Cook-Torrance: 21, He-Torrance: 1516
  - no importance sampling
- Empirical models:
  - behavioral validity: plausibility + features
  - simple to compute: Phong: 5, Blinn: 10
  - importance sampling

Fitting to measurement data



$f_r(\omega',\omega)\cos\theta' = L^{ref} / L^{ir}$

Measurements:

$\omega'_1,\omega_1 \quad F_1$

$\omega'_2,\omega_2 \quad F_2$

….

$\omega'_n,\omega_n \quad F_n$

$f_r(\omega',\omega)\cos\theta' = w(\omega',\omega, p_1,p_2,...p_m)$

e.g.: $p_1=k_d$, $p_2=k_s$, p_3 =shine

Least square estimate:

Find $p_1,p_2,...p_m$ by minimizing:
$E(p_1,...p_m)=\Sigma\,(w(\omega_i',\omega_i, p_1,...,p_m)-F_i)^2$

Non-linear system of equations:
$\partial E/\partial p_1 = 0,\ \partial E/\partial p_2 = 0, \ldots, \partial E/\partial p_m = 0$