Specifying a cube in any dimension

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In three dimensions, it is a relatively simple task to write down a list of the edges of a cube, or more generally, a polyhedron. Specifically, let us label the vertices of the unit cube by

$$\left.\begin{array}{llll} V_0 \doteq (0,0,0) & V_1 \doteq (0,0,1) & V_2 \doteq (0,1,0) & V_3 \doteq (0,1,1) \\ V_4 \doteq (1,0,0) & V_5 \doteq (1,0,1) & V_6 \doteq (1,1,0) & V_7 \doteq (1,1,1) \end{array}\right\} \tag{1}$$

(see figure 1). Note that here we have labeled the vertices such that the binary (base 2) representation of the subscript gives the coordinates of the vertex; e.g., the binary representation of the decimal number 5 is 101, which gives the coordinates of $V_5 = (1,0,1)$. We will use this convenient labeling scheme throughout. Using the above vertices, the edges of the unit cube can thus be described by the list of line segments

$$\left.\begin{array}{llll} \overline{V_0V_1}, & \overline{V_0V_2}, & \overline{V_0V_4}, & \overline{V_1V_3}, \\ \overline{V_2V_3}, & \overline{V_3V_7}, & \overline{V_1V_5}, & \overline{V_4V_5}, \\ \overline{V_5V_7}, & \overline{V_2V_6}, & \overline{V_4V_6}, & \overline{V_6V_7}, \end{array}\right\} \tag{2}$$

as can be seen by inspecting figure 1. Observe that once the coordinates of the vertices $V_0, \ldots, V_7$ have been chosen, we can specify the edges by listing the pairs of subscripts (indices) corresponding to the pairs of vertices forming the endpoints of the edges:

$$\left.\begin{array}{llllll} \langle 0,1\rangle, & \langle 0,2\rangle, & \langle 0,4\rangle, & \langle 1,3\rangle, & \langle 2,3\rangle, & \langle 3,7\rangle, \\ \langle 1,5\rangle, & \langle 4,5\rangle, & \langle 5,7\rangle, & \langle 2,6\rangle, & \langle 4,6\rangle, & \langle 6,7\rangle. \end{array}\right\} \tag{3}$$

In addition to be notationally convenient, this way of representation of the edges is also efficient from a computational point of view: to store an edge,

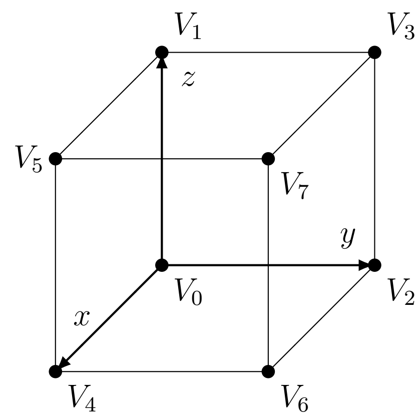

Figure 1: Vertices of the unit cube.

one only needs a *pair* of integers: indices into the "vertex table" (equation (1)); whereas storing the endpoints of an edge amounts to storing *six* floating point values: three values for each component of a three dimensional vertex.

To find a way of writing down a list of edges of higher dimensional cubes, we need to be more systematic in our approach. At the heart of our task is the ability to compute the *boundary* of a cube. For a solid cube, the boundary of the cube is the collection of six square faces that make up the surface of the cube; the boundary of a square is the collection of four line segments (edges) that make up the perimeter of the square; and the boundary of a line segment consists of its two endpoints. In particular, the edges of a cube are the boundaries of its individual square faces; its vertices are the boundaries of its edges.

1 Constructing cubes

The simplest (nontrivial) cube is the one–dimensional cube; i.e., a line segment, or *interval*, in one–dimensional space $\mathbb{R}$. To distinguish a single one–dimensional cube from among all possible ones, we define the **unit one–cube** $\mathcal{C}_1$ to be the interval $[0, 1]$; that is,

$$\mathcal{C}_1 \doteq [0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}. \tag{4}$$

We may form all higher–dimensional cubes inductively using the Cartesian product of intervals: if $n > 1$, then the unit n–cube is the set

$$\mathcal{C}_n \doteq \mathcal{C}_{n-1} \times [0, 1]. \tag{5}$$

That is, we construct higher dimensional cubes from lower dimensional cubes simply by introducing extra coordinates.

For example, the unit two–cube (a square) is constructed from the unit one–cube (a line segment): $\mathcal{C}_2 = \mathcal{C}_1 \times [0,1] = [0,1] \times [0,1]$. In other words, $\mathcal{C}_2$ is the set of all ordered pairs (x_1, x_2) whose components are in $[0,1]$:

$$\mathcal{C}_2 \doteq [0,1] \times [0,1] = \{(x_1, x_2) \mid 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}.$$

Similarly, the unit three–cube (the cube as we normally think of it) is constructed from the unit two–cube (a square): $\mathcal{C}_3 = \mathcal{C}_2 \times [0,1]$. Technically, this means that points of the three cube should are of the form $((x_1, x_2), x_3))$, where $0 \leq x_i \leq 1$ $(i = 1, 2, 3)$; i.e., $(x_1, x_2) \in \mathcal{C}_2$ and $x_3 \in [0,1]$. However, we may make the natural identification of three–tuples of the form $((x_1, x_2), x_3))$ with those of the form (x_1, x_2, x_3) (after all, both are just ordered triples of numbers). Therefore, we may write

$$\begin{aligned}\mathcal{C}_3 &= [0,1] \times [0,1] \times [0,1] \\ &= \{(x_1, x_2, x_3) \mid 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, 0 \leq x_3 \leq 1\}.\end{aligned}$$

And in general, we may regard points in $\mathcal{C}_n$ as n–tuples $(x_1, x_2, \ldots, x_n)$, where $0 \leq x_i \leq 1$ $(i = 0, 1, \ldots, n)$.

2 The boundary calculus

The above inductive procedure for constructing cubes also gives rise to an inductive procedure for computing the boundary of a cube. Indeed, the boundary of the unit one–cube is the set consisting of its endpoints, which we denote by

$$\partial[0,1] = [0] \cup [1]. \tag{6}$$

Here $[x]$ is the set consisting of the single point $x \in \mathbb{R}$, and the symbol $\cup$ denotes the *union* of two sets; so an equivalent restatement of equation (6) would be $\partial[0,1] = \{0, 1\}$.

To extend to cubes of higher dimension, a few more–or–less standard rules from set theory and point set topology are need. Let us enumerate those that are needed. First, the space consisting of a single point has no boundary; in particular,

$$\partial[0] = \emptyset \quad \text{and} \quad \partial[1] = \emptyset. \tag{7}$$

Here $\emptyset$ denotes the *empty set*: the set containing no elements. Second, the boundary of a Cartesian product of *closed* subspaces is given by the so–called *Leibniz rule*:

$$\partial(A \times B) = \partial A \times B \,\cup\, A \times \partial B \tag{8}$$

for closed sets A and B (note that all cubes are closed: they contain their boundary points). Third, the Cartesian product distributes over unions:

$$A \times (B \cup C) = A \times B \,\cup\, A \times C \quad \text{and} \quad (A \cup B) \times C = A \times C \,\cup\, B \times C \tag{9}$$

for any sets A, B, C. The remaining three facts that we will need are elementary facts from set theory, namely

$$A \cup A = A \tag{10}$$

$$A \cup \emptyset = A \tag{11}$$

$$A \times \emptyset = \emptyset \quad \text{and} \quad \emptyset \times A = \emptyset \tag{12}$$

for any set A. The rules (6) through (12) allow us to completely determine the boundary, and consequently the edges, of a cube in any dimension.

Exercise 1. *In point–set topology, one defines the* boundary *of a subspace A of a topological space X as $\partial A \doteq \bar{A} \cap \overline{A^c}$, where $\bar{A}$ is the* closure *of A in X, and A^c is the* complement *of A in X. Show that $\partial X = \emptyset$.*

Exercise 2. *Suppose X, Y are topological spaces. Show that if $A \subseteq X$ and $B \subseteq Y$, then $\partial(A \times B) = \partial A \times \bar{B} \,\cup\, \bar{A} \times \partial B$.*

3 Using the boundary calculus

As a simple example, we determine the edges of the unit two–cube $\mathcal{C}_2$. We compute

$$\begin{aligned}
\partial\mathcal{C}_2 &= \partial([0,1] \times [0,1]) && \text{[eqs. (4),(5)]}\\
&= \partial[0,1] \times [0,1] \,\cup\, [0,1] \times \partial[0,1] && \text{[eq. (8)]}\\
&= ([0] \cup [1]) \times [0,1] \,\cup\, [0,1] \times ([0] \cup [1]) && \text{[eq. (6)]}\\
&= [0] \times [0,1] \,\cup\, [1] \times [0,1] \,\cup\, [0,1] \times [0] \,\cup\, [0,1] \times [1] && \text{[eq. (9)]}
\end{aligned}$$

To interpret the answer as something familiar, we compute the boundary of each edge just computed. For the first edge:

$$\begin{aligned}
\partial([0] \times [0,1]) &= \partial[0] \times [0,1] \cup [0] \times \partial[0,1] && \text{[eq. (8)]} \\
&= \emptyset \times [0,1] \cup [0] \times ([0] \cup [1]) && \text{[eqs. (7),(6)]} \\
&= \emptyset \cup [0] \times [0] \cup [0] \times [1] && \text{[eqs. (12),(9)]} \\
&= [0] \times [0] \cup [0] \times [1] && \text{[eq. (11)].}
\end{aligned}$$

That is, the first edge has vertices $V_0 = (0,0)$ and $V_1 = (0,1)$. In a similar fashion, one finds that the second edge has vertices $V_2 = (1,0)$ and $V_3 = (1,1)$, the third edge has vertices V_0 and V_2, and the fourth edge has vertices V_1 and V_3. Consequently, the edge list is

$$\langle 0,1\rangle, \langle 2,3\rangle, \langle 0,2\rangle, \langle 1,3\rangle, \tag{13}$$

as one would expect using our vertex labeling scheme (see the back face of the cube in figure 1).

Exercise 3. *Verify the entries of the edge list for the unit two–cube given in equation* (13) *using the boundary calculus.*

Let us now briefly outline how our boundary calculus can be used to reproduce the list of edges for the unit three–cube given in equation (3). First, first one computes $\partial\mathcal{C}_3$; the result is

$$\begin{aligned}
\partial\mathcal{C}_3 = [0] \times [0,1] \times [0,1] \cup [1] \times [0,1] \times [0,1] \cup [0,1] \times [0] \times [0,1] \\
\cup [0,1] \times [1] \times [0,1] \cup [0,1] \times [0,1] \times [0] \cup [0,1] \times [0,1] \times [1].
\end{aligned} \tag{14}$$

This gives us the six faces of the unit three–cube. Second, one computes the edges of each of these faces using the same methodology as in the case of the unit two–cube. For example, we compute the edges of the last of the faces in equation (14) to be:

$$\begin{aligned}
\partial([0,1] \times [0,1] \times [1]) = [0] \times [0,1] \times [1] \cup [1] \times [0,1] \times [1] \\
\cup [0,1] \times [0] \times [1] \cup [0,1] \times [1] \times [1];
\end{aligned} \tag{15}$$

the vertices are $V_1 = (0,0,1)$, $V_3 = (0,1,1)$, $V_5 = (1,0,1)$, and $V_7 = (1,1,1)$ and the edges are given by the list

$$\langle 1,3\rangle, \langle 5,7\rangle, \langle 1,5\rangle, \langle 3,7\rangle \tag{16}$$

(these are the edges of the top face of the cube in figure 1). Repeating the this procedure for the other six face reproduces the vertex table and edge list of the unit three–cube in equations (1) and (3).

Exercise 4. *Verify in entirety the vertex table and edge list of the unit three–cube given in equations* (1) *and* (3) *using the boundary calculus.*

We need not restrict ourselve to edges alone in using the boundary calculus; in fact, we can compute all the m–faces of the unit n–cube: the m–cubes contained in the n–cube (the edges are 1–cubes). For example, from equation (15), we find that the last of the 2–faces of the unit three–cube given in equation (14) is given by $\langle 1, 3, 7, 5 \rangle$; that is, it is the face with edges given by equation (16).

Exercise 5. *Using the boundary calculus, find all the* 2*–faces of the unit three–cube.*

Exercise 6. *Using the boundary calculus, find all the edges and* 2*–faces (squares) of the unit four–cube.*

4 References

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