Introduction to quaternions

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The reader may already be familiar with the *complex numbers* $\mathbb{C}$: numbers that can be written in the form $a + bi$, where i is the *imaginary* number with the defining property $i^2 = -1$. The complex numbers are an extension of the real numbers $\mathbb{R}$; i.e., a set of numbers that contains the real numbers. And since there is no real number whose square is a negative number, the containment is proper: $\mathbb{R} \subseteq \mathbb{C}$, but $\mathbb{R} \neq \mathbb{C}$.

The *quaternions* $\mathbb{H}$, on the other hand, form a proper extension of the complex numbers: $\mathbb{C} \subseteq \mathbb{H}$ and $\mathbb{C} \neq \mathbb{H}$. However, in contrast to complex numbers, quaternions are *noncommutative*: $xy \neq yx$ (in general)[1].

1 Definition of quaternions

To define the quaternions, we introduce three new symbols i, j, and k. A **quaternion** then is a number that can be written in the form $a+bi+cj+dk$, where a, b, c, d are real numbers; or in other words,

$$\mathbb{H} \doteq \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}. \tag{1}$$

This suffices to add two quaternions: we simply add term–wise. However, to be able to multiply quaternions, we will need to specify what the products ij, ik, et cetera, should be. Indeed, one stipulates the relations

$$\left.\begin{array}{lll} i^2 = -1, & ij = k, & ik = -j, \\ ji = -k, & j^2 = -1, & jk = i, \\ ki = j, & kj = -i, & k^2 = -1. \end{array}\right\} \tag{2}$$

[1] One can also find a proper extension of the quaternions as well, but a high price must be paid: such numbers will neither be commutative nor *associative*, that is $(xy)z \neq x(yz)$. These numbers are called the *octonians*, and extension game ends here: there is no proper extension of the octonians.

Note that the symbols i, j, k are *skew commutative*: $xy = -yx$ if $x \neq y$ (although this is not true of quaternions in general). By identifying the symbol i used here with the complex number i, we obtain our extension of the complex numbers, $\mathbb{C} \subseteq \mathbb{H}$.

The addition and multiplication of quaternions is completely defined by definition (1) and equations (2), provided we assume the usual rules of algebra and we are careful of the order of multiplication. For example,

$$\begin{aligned}(3+4i-2k)(4j+5k) &= 12j+15k+16ij+20ik-8kj-10k^2 \\ &= 12j+15k+16k-20j+8i+10 \\ &= 10+8i-8j+31k.\end{aligned}$$

It is not difficult to work out general formulas for addition and multiplication of quaternions; however, it is more convenient to express these and other quaternion formulas using the so–called *scalar+vector* notation, which we introduce in the next section.

Exercise 1. *Compute* $(2-5i+3j)(3+4j-k)$.

2 Scalar+vector notation

We call a quaternion in the form $bi+cj+dk$ a **pure imaginary** quaternion. Since a pure imaginary quaternion is effectively a triplet of real numbers, we may identify it with a three–dimensional vector, say $\mathbf{v}$; indeed, the reader may have seen the symbols i, j, k used to denote the unit vectors along the coordinate axes:

$$\mathbf{v} = ai + bj + ck.$$

Thus if $a \in \mathbb{R}$ and $\mathbf{v} = (v_x, v_y, v_z) \in \mathbb{R}^3$, we may write $a + \mathbf{v}$ to denote the quaternion

$$a + \mathbf{v} = a + v_x i + v_y j + v_z k.$$

In scalar+vector notation, the equations for addition and multiplication can be written as

$$(a_1+\mathbf{v}_1)+(a_2+\mathbf{v}_2) = (a_1+a_2)+(\mathbf{v}_1+\mathbf{v}_2) \tag{3}$$

$$(a_1+\mathbf{v}_1)(a_2+\mathbf{v}_2) = (a_1a_2-\mathbf{v}_1\cdot\mathbf{v}_2)+(a_1\mathbf{v}_2+a_2\mathbf{v}_1+\mathbf{v}_1\times\mathbf{v}_2), \tag{4}$$

where $\cdot$ and $\times$ are the standard dot and cross products of three–dimensional vectors. Thus with scalar+vector notation, quaternion arithmetic computations are essentially reduced to three–dimensional vector arithmetic.

Exercise 2. *Verify equations* (3) *and* (4) *using the relations* (2).

3 Quaternion conjugate

The **conjugate** of a quaternion is obtained by negating its imaginary part (this is analogous to the complex conjugate):

$$\overline{(a+bi+cj+dk)} \doteq a-bi-cj-dk.$$

Or in scalar+vector notation, $\overline{(a+\mathbf{v})} = a-\mathbf{v}$. The usefulness of this operation comes from the following formula, which equates the product of a quaternion and its conjugate with the square of the length of the quaternion, viewed as a vector in four–dimensions:

$$(a+bi+cj+dk)\overline{(a+bi+cj+dk)} = a^2+b^2+c^2+d^2. \tag{5}$$

Or stated more succinctly, if $q \in \mathbb{H}$, then $q\bar{q} = |q|^2$, where $|\cdot|$ denotes four–dimensional vector length.

Exercise 3. *Verify equation* (5) *using equation* (4).

Armed with formula (5), we can obtain a formula for the inverse of a quaternion: the equation $q\bar{q} = |q|^2$ can be inverted to give

$$q^{-1} = |q|^{-2}\bar{q}, \tag{6}$$

provided $|q| \neq 0$ (or equivalently, $q \neq 0$). For example,

$$(1+3j-2k)^{-1} = \frac{1}{1^2+3^2+(-2)^2}(1-3j+2k) = \tfrac{1}{14} - \tfrac{3}{14}j + \tfrac{1}{7}k.$$

As a consequence, division is also defined for quaternions, although once again, we must be careful of the order of division: pq^{-1} and $q^{-1}p$ will result in different quaterions.

Exercise 4. *Compute* $(2-5i+2j+k)^{-1}$.

Exercise 5. *Compute* $(1+3j)^{-1}(2+i-j+k)$ *and* $(2+i-j+k)(1+3j)^{-1}$.

Another consequence of equation (5) is the following, which gives an identity for the inner product of two quaternions (viewed as four–dimensional vectors) in terms of quaternion arithmetic:

$$p \cdot q = \tfrac{1}{2}(p\bar{q} + q\bar{p}). \tag{7}$$

Here, if $p = a_1 + b_1 i + c_1 j + d_1 k$ and $q = a_2 + b_2 i + c_2 j + d_2 k$, then $p \cdot q = a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2$.

Exercise 6. *Verify equation* (7) *using the identity* $q\bar{q} = |q|^2$ *and the* polarization identity *for four–dimensional vectors* $\mathbf{u}, \mathbf{v}$*:* $\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}(|\mathbf{u}+\mathbf{v}|^2 - |\mathbf{u}-\mathbf{v}|^2)$.

4 Amusing identities

Here are some useful, or at least interesting, formulas involving quaternions. We use scalar+vector notation exclusively.

- The *commutator* $[p, q] \doteq pq - qp$ of two quaternions $p, q \in \mathbb{H}$ can be thought of measuring the extent of noncommutativity of quaternion multiplication. In terms of scalar+vector notation, we have

  $$(a_1 + \mathbf{v}_1)(a_2 + \mathbf{v}_2) - (a_2 + \mathbf{v}_2)(a_1 + \mathbf{v}_1) = 2\mathbf{v}_1 \times \mathbf{v}_2. \tag{8}$$

- The following indicates the *Clifford algebra* structure of the quaternions:

  $$\mathbf{v}^2 = -|\mathbf{v}|^2. \tag{9}$$

  In particular, if $|\mathbf{v}| = 1$, then $\mathbf{v}^2 = -1$ (Clifford algebras, among other things, are used to construct *spin representations* of the orthogonal groups).

- Reflection of a three–dimensional vector through a plane can be obtained through quaternion arithmetic:

  $$\mathbf{vuv} = |\mathbf{v}|^2\mathbf{u} - 2(\mathbf{u} \cdot \mathbf{v})\mathbf{v}, \tag{10}$$

  so that if $|\mathbf{v}| = 1$, this gives the formula for the reflection of $\mathbf{u}$ through the plane orthogonal to the (unit) vector $\mathbf{v}$. The operation on the left is sometimes called *Clifford conjugation.*

- Here is the quaternion version of the *De Moivre formula* from complex numbers: if $|\mathbf{v}| = 1$, then

$$(\cos\theta + \sin\theta\mathbf{v})^n = \cos n\theta + \sin n\theta\mathbf{v} \tag{11}$$

for all integers n.

- Three–dimensional rotations can also be obtained through quaternion arithmetic: if $|\mathbf{v}| = 1$ and $q = \cos\theta + \sin\theta\mathbf{v}$ (so that $|q| = 1$),

$$q\mathbf{u}\bar{q} = (\mathbf{v}\cdot\mathbf{u})(1 - \cos 2\theta)\mathbf{v} + \cos 2\theta\mathbf{u} + \sin 2\theta\mathbf{v}\times\mathbf{u}, \tag{12}$$

which is the formula for the counterclockwise rotation of the three–dimensional vector $\mathbf{u}$ about the three–dimensional vector $\mathbf{v}$ by 2θ radians.

Exercise 7. *Verify equations* (8)–(12).

5 References

- *Quaternions and reflections*, by H.S.M. Coxeter, The American Mathematical Monthly, Volume 53, Number 3 (March 1946), 136–146.