Cosets and group actions

jason hanson *

1 Cosets

Let G be a group with binary operation $\cdot : G \times G \to G$. Recall that by convention, we suppress the explicit use of the operator $\cdot$; thus if $a, b \in G$, then we write ab instead of $a \cdot b$. Also recall that e denotes the identity element of G, and g^{-1} denotes the inverse of g in G. Furthermore, recall that H is a *subgroup* of G if H is both a subset of G and a group in and of itself; that is, H is closed under group multiplication: if $c, d \in H$, then $cd \in H$. By way of notation, we write $H \leq G$ whenever H is a subgroup of G.

Definition 1. *Suppose G, H are groups with $H \leq G$. If $g \in G$, then the* **(left) coset** *of H by g is the subset $gH \doteq \{gh \mid h \in H\} \subseteq G$. The collection of all possible left cosets of H by elements in G is denoted by G/H; i.e., $G/H \doteq \{gH \mid g \in G\}$.*

For example, suppose that $G = S_3$, the group of permutations on three letters. Explicitly in terms of cycle notation:

$$G = \{(), (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}.$$

Let H be the subgroup generated by the permutation $(1,2)$; that is,

$$H = \{(), (1,2)\}$$

(which is isomorphic to C_2, the cyclic group of order 2). Then

$$(1,3)H = \{(1,3), (1,2,3)\}.$$

Observe that $(1,2)H = \{(1,2), ()\} = eH = H$, so that not all cosets are distinct. Distinct cosets, however, are necessarily disjoint.

Exercise 1. *Show that* $gH = g'H$ *if and only if* $g^{-1}g' \in H$.

Exercise 2. *Show that if* $H \leq G$ *and* $g, g' \in G$, *then either* $gH = g'H$ *or* $gH \cap g'H = \emptyset$.

It follows that G/H gives a *partition* of G: a disjoint collection of subsets of G whose union is all of G. In the case when $G = S_3$ and H is as above, we have

$$G/H = \{eH, (1,3)H, (2,3)H\}.$$

Since $(2,3)H = \{(2,3), (1,3,2)\}$ ($eH = H$ and $(1,3)H$ were given previously), we see that G/H indeed gives a partition of S_3.

1.1 Normal subgroups

Definition 2. *Given* $H \leq G$, *the* **conjugate** *of* H *by* $g \in G$ *is the subset* $gHg^{-1} \doteq \{ghg^{-1} \mid h \in H\}$.

Thus if $G = S_3$ and $H = \{(), (1,2)\}$ (as in our previous example), then $(1,3)H(1,3)^{-1} = (1,3)H(1,3) = \{(), (2,3)\}$, since $(1,3)(1,2)(1,3) = (2,3)$. One often makes the abbreviation $H^g \doteq gHg^{-1}$ for the conjugate of H by g.

Exercise 3. *Prove: if* $H \leq G$, *then for any* $g \in G$, H^g *is a subgroup of* G.

Definition 3. *A subgroup* $N \leq G$ *is* **normal** *if* $N^g = N$ *for all* $g \in G$. *In this case, we write* $N \trianglelefteq G$.

Since $H^{(1,3)} = \{(), (2,3)\} \neq H$, the subgroup H of S_3 is *not* a normal subgroup. However, the subgroup $N \doteq \{(), (1,2,3), (1,3,2)\}$ *is* a normal subgroup of S_3, as the reader will verify.

Exercise 4. *Suppose* $N \trianglelefteq G$. *Show that the multiplication rule in* G/N *given by* $(gN)(g'N) \doteq (gg'N)$ *is well–defined; that is, if* $gN = \tilde{g}N$ *and* $g'N = \tilde{g}'N$, *then* $(gg')N = (\tilde{g}\tilde{g}')N$.

Exercise 5. *Show that if* $N \trianglelefteq G$, *then* G/N *is a group under the above group multiplication rule.*

In particular, eN is the identity element in G/N; and $(gN)^{-1} = g^{-1}N$. Moreover if $|A|$ denotes the size of the set A, then $|gN| = |N|$ for all $g \in G$; thus G/N is a group of order $|G/N| = |G|/|N|$.

As an example, if $G = S_3$ and $N \doteq \{(), (1,2,3), (1,3,2)\}$ (as before), then G/N is a group of order 2 (hence isomorphic to C_2), the elements of which are $G/N = \{eN, (1,2)N\}$.

1.2 Exact sequences

Recall that a map $h : G \to G'$ between groups G, G' is a *homomorphism* if it preserves group multiplication: $h(g_1 g_2) = h(g_1)h(g_2)$ for all $g_1, g_2 \in G$.

Exercise 6. *Prove that the image of a group homomorphism is a subgroup of its range:* $h(G) \leq G'$.

Exercise 7. *Let* 1 *denote the* trivial group*: the group of order 1 consisting of only the identity element* e*. Show that the maps* $i : 1 \to G$*, where* $i(e) \doteq e$*, and* $j : G \to 1$*, where* $j(G) \doteq e$*, are group homomorphisms.*

Exercise 8. *Prove that if* $N \trianglelefteq G$*, then the* quotient map $q : G \to G/N$*, where* $q(g) \doteq gN$*, is a group homomorphism.*

Definition 4. *The* **kernel** *of a group homomorphism* $h : G \to G'$ *is the set of all elements in* G *that map to the identity:* $\ker(h) \doteq \{g \in G \mid h(g) = e\}$.

Exercise 9. *Show that the kernel is a normal subgroup:* $\ker(h) \trianglelefteq G$.

Exercise 10. *Prove that a homomorphism is injective (one–to–one) if and only if its kernel is trivial:* $\ker(h) = 1$.

Definition 5. *Suppose that* $G_1, \ldots, G_n$ *are groups, and that* $h_i : G_i \to G_{i+1}$ $(i = 1, \ldots, n-1)$ *are group homomorphisms. The sequence*

$$G_1 \xrightarrow{h_1} G_2 \xrightarrow{h_2} \ldots \xrightarrow{n_{n-2}} G_{n-1} \xrightarrow{h_{n-1}} G_n \tag{1}$$

is **exact at** G_i *if* $\mathrm{im}(h_{i-1}) = \ker(h_i)$*. The sequence* (1) *is* **exact** *if it is exact at* G_i *for all* $i = 2, \ldots, n-1$.

In the above definition, $\mathrm{im}(h_i)$ denotes the *image* of the map h_i; that is, $\mathrm{im}(h_i) = h_i(G_i)$.

If $N \trianglelefteq G$, the inclusion map $incl : N \to G$ ($incl(n) \doteq n$) is a priori a group homomorphism, so we have the sequence:

$$1 \xrightarrow{i} N \xrightarrow{incl} G \xrightarrow{q} G/N \xrightarrow{j} 1. \tag{2}$$

The maps i and j are as in exercise 7, and q is the quotient map as in exercise 8. This sequence is exact at N, since *incl* is injective; whence $\mathrm{im}(i) = \{e\} = \ker(incl)$. It is exact at G, since the kernel of q is just N; i.e., $\mathrm{im}(incl) = N = \ker(q)$. Moreover, the sequence is also exact at G/N, since the quotient map is surjective: $\mathrm{im}(q) = G/N = \ker(j)$. Therefore, the sequence (2) is an exact sequence.

Definition 6. *If* $1 \to N \to G \to K \to 1$ *is an exact sequence, we say that* G *is an* **extension** *of* N *by* K*.*

Note that we necessarily have that $N \trianglelefteq G$, and the map $G \to K$ is surjective.

Exercise 11. *Prove that if* G *is an extension of* N *by* K*, then* K *is isomorphic to* G/N*.*

Thus the exact sequence $1 \to N \to G \to K \to 1$ is equivalent to the sequence (2).

Exercise 12. *Show that for groups* N *and* K*, the sequence* $1 \to N \to N \times K \to K \to 1$ *is exact, where* $N \to N \times K$ *is the map* $n \mapsto (n, e)$*, and* $N \times K \to K$ *is the map* $(n, k) \mapsto k$*.*

Group extensions are not unique in general. For example, in the case where $N \doteq \{(), (1,2,3), (1,3,2)\}$, we have the exact sequence

$$1 \to N \to S_3 \to S_3/N \to 1,$$

where $N \to S_3$ is the inclusion, and $S_3 \to S_3/N$ is the quotient map; as we have already seen, S_3/N is isomorphic to C_2. On the other hand, we also have the exact sequence

$$1 \to N \to N \times C_2 \to C_2 \to 1$$

(as in the previous exercise). However, S_3 is not isomorphic to the direct product $N \times C_2$, since S_3 is not Abelian.

2 Group actions

Definition 7. *Let* G *be a group and* X *a set. A map* $\alpha : G \times X \to X$ *is a* **group action** *if for all* $x \in X$*, we have (1)* $\alpha(e, x) = x$*, and (2)* $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$ *for all* $g_1, g_2 \in G$*.*

In this case, we also say that G *acts on* X. One usually omits explicit reference to the map α and simply writes gx in lieu of $\alpha(g, x)$. The properties of a group action are then $ex = x$ and $g_1(g_2 x) = (g_1 g_2)x$.

A simple example of a group action is C_2 acting on $\mathbb{R}$ by negation. That is, $\alpha : C_2 \times \mathbb{R} \to \mathbb{R}$ is given by $\alpha(e, r) \doteq r$ and $\alpha(\gamma, r) \doteq -r$, where γ is the generator of C_2 (so that $\gamma^2 = e$); or, omitting α from the notation: $er = r$ and $\gamma r = -r$.

Exercise 13. *Let* $C : G \times G \to G$ *be the map* $C(g, g') \doteq gg'g^{-1}$*. Show that* C *is a group action. In this case, we say that* G acts on itself by conjugation.

Exercise 14. *Suppose* $H \leq G$*. Define* $L : G \times G/H$ *by* $L(g', gH) \doteq g'gH$*. Verify that* L *is a group action.*

Definition 8. *Suppose that* G *acts on* X*, and that* $x \in X$*. The* **orbit** *of* x *is the set* $Gx \doteq \{gx \mid g \in G\} \subseteq X$*, and the* **stabilizer** *of* x *is the set* $G_x \doteq \{g \in G \mid gx = x\} \subseteq G$*.*

Exercise 15. *Show that the stabilizer is a subgroup:* $G_x \leq G$*.*

Exercise 16. *Prove that* $G_{gx} = (G_x)^g$*.*

Exercise 17. *Show that the map* $Gx \to G/G_x$*, given by* $gx \mapsto gG_x$*, is well–defined and a bijection (one–to–one and onto).*

Exercise 18. *Suppose* $x, y \in X$*. Prove that either* $Gx = Gy$ *or* $Gx \cap Gy = \emptyset$*.*

As a consequence, the distinct of orbits of X form a partition of X. In the case where G acts on itself by conjugation, the orbit of $g \in G$ is called the *conjugacy class* of g in G.

3 References

- *Contemporary abstract algebra*, sixth edition, by Joseph A. Gallian; published by Houghton Mifflin Company, 2005; ISBN: 0618514716.

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