

Projecting from four to three dimensions

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1 Projecting into three dimensions

Although there are many ways to project from a higher dimension to a lower dimension, the most natural is perhaps an orthogonal projection. Specifically, for projecting from four-dimensional space down to three-dimensional space, we identify three-dimensional space with a hyperplane (three-dimensional subspace) in four-dimensional space, and orthogonally project four-dimensional points onto this hyperplane.

Specifically, choose three linearly independent vectors in four-dimensional space, say \mathbf{t} , \mathbf{u} , and \mathbf{v} , to define the hyperplane. We then use the map

$$i(x, y, z) \doteq x\mathbf{t} + y\mathbf{u} + z\mathbf{v}, \quad (1)$$

to identify points (x, y, z) in three-dimensional space with points $i(x, y, z)$ in the hyperplane spanned by $\mathbf{t}, \mathbf{u}, \mathbf{v}$. The orthogonal projection of a point (a, b, c, d) in four-dimensional space onto the hyperplane is obtained as follows. Let \mathbf{s} be a vector orthogonal to the hyperplane. Then the point $\mathbf{p} = (a, b, c, d)$ can be written uniquely as a linear combination

$$\mathbf{p} = w\mathbf{s} + x\mathbf{t} + y\mathbf{u} + z\mathbf{v}, \quad (2)$$

for some real numbers w, x, y, z . Since $\mathbf{t}, \mathbf{u}, \mathbf{v}$ are assumed to be orthogonal to \mathbf{s} , we obtain the three equations

$$\begin{aligned} x(\mathbf{t} \cdot \mathbf{t}) + y(\mathbf{t} \cdot \mathbf{u}) + z(\mathbf{t} \cdot \mathbf{v}) &= \mathbf{p} \cdot \mathbf{t} \\ x(\mathbf{t} \cdot \mathbf{u}) + y(\mathbf{u} \cdot \mathbf{u}) + z(\mathbf{u} \cdot \mathbf{v}) &= \mathbf{p} \cdot \mathbf{u} \\ x(\mathbf{t} \cdot \mathbf{v}) + y(\mathbf{u} \cdot \mathbf{v}) + z(\mathbf{v} \cdot \mathbf{v}) &= \mathbf{p} \cdot \mathbf{v}, \end{aligned}$$

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where \cdot is the Euclidean usual inner product on four-dimensional space: $(a, b, c, d) \cdot (a', b', c', d') \doteq aa' + bb' + cc' + dd'$. The above set of equations can be cast into a matrix equation:

$$\begin{pmatrix} \mathbf{t} \cdot \mathbf{t} & \mathbf{t} \cdot \mathbf{u} & \mathbf{t} \cdot \mathbf{v} \\ \mathbf{t} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{t} \cdot \mathbf{v} & \mathbf{u} \cdot \mathbf{v} & \mathbf{v} \cdot \mathbf{v} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mathbf{p} \cdot \mathbf{t} \\ \mathbf{p} \cdot \mathbf{u} \\ \mathbf{p} \cdot \mathbf{v} \end{pmatrix}. \quad (3)$$

By inverting this equation, we obtain the orthogonal projection of the point \mathbf{p} in four-dimensional space onto the point $i(x, y, z)$ in the plane space spanned by the four-dimensional vectors \mathbf{t} , \mathbf{u} , and \mathbf{v} :

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mathbf{t} \cdot \mathbf{t} & \mathbf{t} \cdot \mathbf{u} & \mathbf{t} \cdot \mathbf{v} \\ \mathbf{t} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{t} \cdot \mathbf{v} & \mathbf{u} \cdot \mathbf{v} & \mathbf{v} \cdot \mathbf{v} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{p} \cdot \mathbf{t} \\ \mathbf{p} \cdot \mathbf{u} \\ \mathbf{p} \cdot \mathbf{v} \end{pmatrix}. \quad (4)$$

Note that because \mathbf{t} , \mathbf{u} , and \mathbf{v} are assumed to be linearly independent, the 3×3 matrix in this equation is indeed invertible; in the special case when the basis vectors are *orthonormal*, the matrix is in fact the identity.