

The Mathematicians torus

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1 Standard torus in three dimensions

Let us call the torus obtained by rotating the circle in the xz -plane, of radius r and centered at $(R, 0, 0)$, about the z -axis the **standard torus** with major radius R and minor radius r (we assume that $0 < r < R$). That is, points (x, y, z) on the standard torus satisfy the equation:

$$(x^2 + y^2 + z^2 + R^2 - r^2)^2 - 4R^2(x^2 + y^2) = 0. \quad (1)$$

Equation (1) gives an *implicit* representation of the standard torus. An equivalent *parametric* representation is:

$$x = (R + r \cos \theta) \cos \phi, \quad y = (R + r \cos \theta) \sin \phi, \quad z = r \sin \theta, \quad (2)$$

for $0 \leq \theta < 2\pi$ and $0 \leq \phi < 2\pi$.

Exercise 1. Verify equation (1) by replacing the symbol x in the equation for the circle $(x - R)^2 + z^2 - r^2 = 0$ in the xz -plane with $\rho \doteq \sqrt{x^2 + y^2}$.

Exercise 2. Show that equation (2) is a solution to the implicit equation of a torus, equation (1).

Exercise 3. Show that every point (x, y, z) that satisfies equation (1) can be written in the form of equation (2).

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2 The Mathematicians torus

From a Mathematicians point of view, a torus is the Cartesian product of two circles: $S^1 \times S^1$, which we dub the **Mathematicians torus**. Since a circle S^1 is naturally embedded in the two-dimensional plane: it is the set of all points (x, y) with

$$x^2 + y^2 - 1 = 0, \quad (3)$$

the torus $S^1 \times S^1$ is most naturally embedded in four-dimensional space: the set of all points (w, x, y, z) with

$$w^2 + x^2 - 1 = 0 \quad \text{and} \quad y^2 + z^2 - 1 = 0. \quad (4)$$

We can write down an explicit equivalence (homeomorphism, actually diffeomorphism in this case) between $S^1 \times S^1$ in four-dimensional space and the standard torus in three-dimensional space:

$$f(a, b, c, d) \doteq ((R + rc)a, (R + rc)b, rd). \quad (5)$$

I.e., (a, b, c, d) is a point on $S^1 \times S^1$ if and only if $f(a, b, c, d)$ is a point the standard torus. The inverse of f is given by

$$g(x, y, z) \doteq \left(\frac{Rx}{\sqrt{x^2 + y^2}}, \frac{Ry}{\sqrt{x^2 + y^2}}, \sqrt{x^2 + y^2} - R, z \right). \quad (6)$$

That is, (x, y, z) is a point on the standard torus if and only if $g(x, y, z)$ is a point on $S^1 \times S^1$ in four-dimensional space.

Exercise 4. *If (a, b, c, d) is such that $a^2 + b^2 = 1$ and $c^2 + d^2 = 1$, show that equation (5) gives a point $(x, y, z) = f(a, b, c, d)$ that solves equation (1).*

Exercise 5. *If (x, y, z) is a solution to equation (1), show that equation (6) gives a point $(a, b, c, d) = g(x, y, z)$ such that $a^2 + b^2 = 1$ and $c^2 + d^2 = 1$.*

Exercise 6. *Show that f and g in equations (5) and (6) are inverse to each other (as maps between the standard torus in three-dimensional space and $S^1 \times S^1$ in four-dimensional space).*

Exercise 7. *Show that f and g are diffeomorphisms between the standard torus and $S^1 \times S^1$.*

Exercise 8. *Show that f (as a map from $S^1 \times S^1$ to the standard torus) is not a conformal: it does not preserve angles between vectors.*

3 References

For topology and differentiable manifolds (needed to answer some of exercise questions), see the following.

- *Topology*, second edition, by James R. Munkres; published by Prentice Hall, 1999; ISBN: 0131816292.
- *An introduction to differentiable manifolds and Riemannian geometry*, revised edition, by William M. Boothby; published by Elsevier Science and Technology, 2002; ISBN: 0121160513.
- *Manifolds, tensor analysis, and applications*, second edition, by Ralph Abraham, Jerrold E. Marsden, and Tudor Ratiu; published by Springer-Verlag, 1988; ISBN: 0387967907.