QUIZ 6

(Math 200-Section B)

1. Find the Maclaurin polynomials P_0, P_1, P_2 for $f(x) = e^{2x} - 3x + 1$. (4 pts)
   (*Note*: The Maclaurin polynomials are the Taylor polynomials at $x_0 = 0$)

$$f(x) = e^{2x} - 3x + 1 \qquad f(0) = e^0 - 0 + 1 = 2$$
$$f'(x) = 2e^{2x} - 3 \qquad f'(0) = 2e^0 - 3 = -1$$
$$f''(x) = 4e^{2x} \qquad f''(0) = 4e^0 = 4$$

$$P_0 = f(0) = 2$$
$$P_1 = f(0) + f'(0) \cdot x = 2 - x$$
$$P_2 = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} \cdot x^2 = 2 - x + 2x^2$$

2. Expand $f(x) = 2x^3 - 4x^2 + 3x + 1$ in powers of $(x-1)$. What do you observe after the 3^{th} derivative? Explain why is this necessary to happen. (4 pts)

$$f(x) = 1 + 3x - 4x^2 + 2x^3$$
$$f'(x) = 3 - 8x + 6x^2 \qquad f(1) = 1 + 3 - 4 + 2 = 2$$
$$f''(x) = -8 + 12x \qquad f'(1) = 3 - 8 + 6 = 1$$
$$f^{(3)}(x) = 12 \qquad f''(1) = -8 + 12 = 4$$
$$f^{(4)}(x) = 0 \qquad f'''(1) = 12$$
$$f^{(n)}(x) = 0 \quad \forall n \geq 4$$

$$f(x) = f(1) + f'(1) \cdot (x-1) + \frac{f''(1)}{2!} \cdot (x-1)^2 + \frac{f^{(3)}(1)}{3!} \cdot (x-1)^3 + \frac{f^{(4)}(1)}{4!} \cdot (x-1)^4 + \ldots$$
$$= 2 + (x-1) + \frac{4}{2}(x-1)^2 + \frac{12}{6}(x-1)^3 + 0 + 0 + \ldots$$
$$= 2(x-1)^0 + 1(x-1)^1 + 2(x-1)^2 + 2(x-1)^3$$

After the 3rd derivative all further derivatives are zero and thus all further polynomials in the expansion are zero. This is necessary because the Taylor expansion to the third degree already matches *f*(*x*) exactly, i.e.,
$2(x-1)^0 + 1(x-1)^1 + 2(x-1)^2 + 2(x-1)^3 = 2x^3 - 4x^2 + 3x + 1 = f(x)$.

3. Find the Taylor series of $f(x)=\frac{1}{x^2}$ at $x_0=1$. Make sure you include the n^{th} - term of the series. (4 pts)

$f(x)=x^{-2}$ $\quad f(1)=1$

$f'(x)=-2x^{-3}$ $\quad f'(1)=-2$

$f''(x)=6x^{-4}$ $\quad f''(1)=6$

$f^{(3)}(x)=-24x^{-5}$ $\quad f^{(3)}(1)=-24$

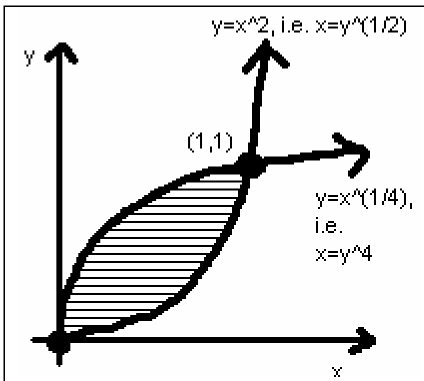
Now, note that $f^{(n)}(x)=(-1)^n\cdot(n+1)!\cdot x^{-2-n}$ and that $f^{(n)}(1)=(-1)^n\cdot(n+1)!$.
So the Taylor series of *f* is

$$f(1)+f'(1)\cdot(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f^{(3)}(1)}{3!}(x-1)^3+\ldots+\frac{f^{(n)}(1)}{n!}(x-1)^n+\ldots$$

$$=1+(-2)(x-1)+\frac{6}{2!}(x-1)^2+\frac{-24}{3!}(x-1)^3+\ldots+\frac{(-1)^n\cdot(n+1)!}{n!}(x-1)^n+\ldots$$

$$=1+(-2)(x-1)+3(x-1)^2+(-4)(x-1)^3+\ldots+(-1)^n(n+1)(x-1)^n+\ldots$$

4. Evaluate $\iint_R (x^{1/2}-y^2)\,dxdy$, where R is the region in the first quadrant bounded by the curves $y=x^2$ and $y=x^{1/4}$. (4 pts)
(*Recall*: The graph of $y=x^{1/4}$ is similar to $y=x^{1/2}=\sqrt{x}$)



$$\int_0^1\int_{y^4}^{y^{1/2}}(x^{1/2}-y^2)\,dxdy$$

$$=\int_0^1\left(\frac{2}{3}x^{3/2}-y^2x\right)\Bigg|_{x=y^4}^{x=y^{1/2}}dy$$

$$=\int_0^1\left(\frac{2}{3}(y^{1/2})^{3/2}-y^2(y^{1/2})-\frac{2}{3}(y^4)^{3/2}+y^2(y^4)\right)dy$$

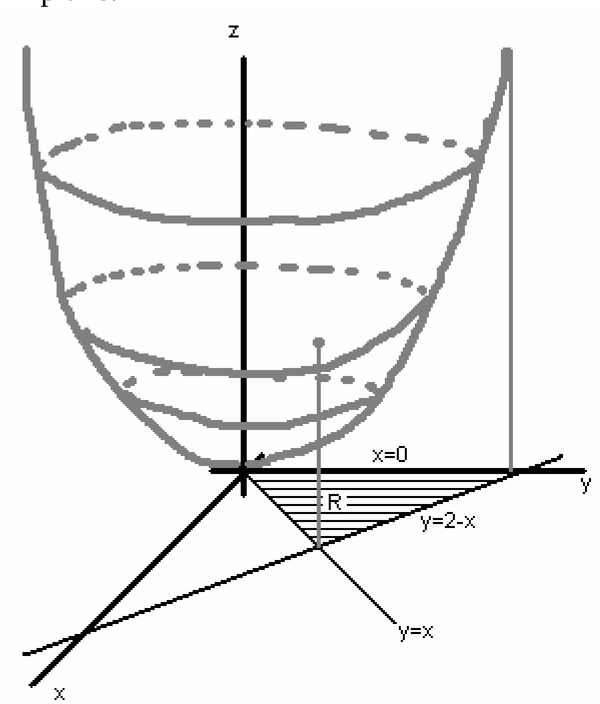
$$=\int_0^1\left(\frac{2}{3}y^{3/4}-y^{5/2}-\frac{2}{3}y^6+y^6\right)dy$$

$$=\int_0^1\left(\frac{1}{3}y^6-y^{5/2}+\frac{2}{3}y^{3/4}\right)dy$$

$$=\left(\frac{1}{21}y^7-\frac{2}{7}y^{7/2}+\frac{8}{21}y^{7/4}\right)\Bigg|_{y=0}^{y=1}$$

$$=\frac{1}{21}-\frac{2}{7}+\frac{8}{21}=\frac{1}{7}$$

5. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x, x = 0$ and $x + y = 2$ in the *xy*-plane. (4 pts)



$$\iint_R z\,dxdy = \int_0^1 \int_x^{2-x} (x^2 + y^2)\,dydx$$

$$= \int_0^1 \left(x^2 y + \frac{1}{3} y^3 \right)\Bigg|_{y=x}^{y=2-x} dx$$

$$= \int_0^1 \left(x^2 (2-x) + \frac{1}{3}(2-x)^3 - x^2 (x) - \frac{1}{3}(x)^3 \right) dx$$

$$= \int_0^1 \left(2x^2 - x^3 + \frac{1}{3}(2-x)(4-4x+x^2) - x^3 - \frac{1}{3}x^3 \right) dx$$

$$= \int_0^1 \left(\frac{1}{3}(8 - 8x + 2x^2 - 4x + 4x^2 - x^3) + 2x^2 - \frac{7}{3}x^3 \right) dx$$

$$= \int_0^1 \left(\frac{8}{3} - 4x + 2x^2 - \frac{1}{3}x^3 + 2x^2 - \frac{7}{3}x^3 \right) dx$$

$$= \int_0^1 \left(\frac{8}{3} - 4x + 4x^2 - \frac{8}{3}x^3 \right) dx$$

$$= \left(\frac{8}{3}x - 2x^2 + \frac{4}{3}x^3 - \frac{2}{3}x^4 \right)\Bigg|_0^1$$

$$= \frac{8}{3} - \frac{6}{3} + \frac{4}{3} - \frac{2}{3} = \frac{4}{3}$$