Key

Midterm Exam

Math 200-Section A
(Fall 2005)

Solve the following problems. Show all your work in the space under each problem.

1. Find the equation of the plane that passes through the point P = (1, -2, -1) and is perpendicular to the line of intersection of the planes $x+2y-z=2$ and $x-y+z=1$. (2 pts)

The vector $\bar{u} = \bar{n}_1 \times \bar{n}_2$, where $\bar{n}_1 = (1,2,-1)$ and $\bar{n}_2 = (1,-1,1)$ is // to the line.

$$\bar{u} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} = i - 2j - 3k$$

The equ of the plane is : $1(x-1) - 2(y+2) - 3(z+1) = 0$

$$\boxed{x - 2y - 3z = 8}$$

2. Find the length of the curve $\mathbf{r}(t) = (2t)i + (2t)j + (3-t)k$ from (0, 0, 3) to (2, 2, 2). (*Note*: Make sure you write the correct limits during the calculation of the integral) (2 pts)

$$\left.\begin{aligned} 0i + 0j + 3k &= (2t)i + (2t)j + (3-t)k \\ 2i + 2j + 2k &= (2t)i + (2t)j + (3-t)k \end{aligned}\right\} \begin{aligned} &\Rightarrow t = 0 \\ &\Rightarrow t = 1 \end{aligned}$$

$$s = \int_0^1 |r'(t)|\, dt = \int_0^1 |2i + 2j - k|\, dt = \int_0^1 \sqrt{4+4+1}\, dt = \int_0^1 \sqrt{9}\, dt$$

$$= \int_0^1 3\, dt$$

$$= [3t]_0^1$$

$$= \boxed{3}$$

3. Given $\mathbf{r}(t) = (3\sin t)i + (3\cos t)j + (4t)k$, show that $4\kappa + 3\tau = 0$, where κ and τ are the curvature and torsion, respectively, of the curve $\mathbf{r}(t)$. (2 pts)

(*Hint*: Use the formulas $\kappa = \dfrac{|v \times a|}{|v|^3}$ and $\tau = \dfrac{\det(\dot{x}, \dot{y}, \dot{z}; \ddot{x}, \ddot{y}, \ddot{z}; \dddot{x}, \dddot{y}, \dddot{z})}{|v \times a|^2}$)

$v = 3\cos t\, i - 3\sin t\, j + 4k$, $a = -3\sin t\, i - 3\cos t\, j + 0k$

$$v \times a = \begin{vmatrix} i & j & k \\ 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \end{vmatrix} = 12\cos t\, i - 12\sin t\, j + (-9\cos^2 t - 9\sin^2 t)k$$

$$= 12\cos t\, i - 12\sin t\, j - 9k$$

$$|v \times a| = \sqrt{144\cos^2 t + 144\sin^2 t + 81} = \sqrt{225} = 15$$

$$|v| = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = \sqrt{9+16} = \sqrt{25} = 5$$

So, $\kappa = \dfrac{15}{5^3} = \dfrac{3 \cdot \cancel{5}}{\cancel{5} \cdot 5^2} = \dfrac{3}{25}$

Now, $\det(\dot{x}, \dot{y}, \dot{z}; \ddot{x}, \ddot{y}, \ddot{z}; \dddot{x}, \dddot{y}, \dddot{z}) = \begin{vmatrix} 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \\ -3\cos t & 3\sin t & 0 \end{vmatrix} = 4(-9\sin^2 t - 9\cos^2 t)$

$$= 4(-9)$$

$$= -36$$

So, $\tau = \dfrac{-36}{15^2} = \dfrac{-4 \cdot \cancel{9}}{(\cancel{3} \cdot 5)(\cancel{3} \cdot 5)} = \dfrac{-4}{25}$

Therefore, $4\kappa + 3\tau = 4(3/25) + 3(-4/25) = 12/25 - 12/25 = 0$.

4. (a) Calculate the limit: $\lim_{(x,y)\to(0,\frac{\pi}{2})} \frac{xy-1+3\sin y}{\cos x+1}$ (4 pts)

$$\lim_{(x,y)\to(0,\pi/2)} \frac{xy-1+3\sin y}{\cos x+1} = \frac{0\cdot\pi/2 - 1 + 3\sin\pi/2}{\cos 0 + 1} = \frac{0-1+3}{1+1} = \frac{2}{2} = \boxed{1}$$

(b) Let $f(x) = \frac{3xy^2}{x^3+y^3}$. Show that $f(x)$ has no limit as $(x,y)\to(0,0)$.

Consider the paths: $y = kx$, $k \in \mathbb{R}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} \frac{3xy^2}{x^3+y^3} = \lim_{(x,y)\to(0,0)} \frac{3x(k^2x^2)}{x^3+k^3x^3} = \lim_{(x,y)\to(0,0)} \frac{3\cancel{x^3}k^2}{\cancel{x^3}(1+k^3)} = \lim_{(x,y)\to(0,0)} \frac{3k^2}{1+k^3} = \frac{3k^2}{1+k^3}$$

Hence, for different k we get different values for the limit, ie f has no limit.

5. Find the directional derivative of $f(x,y,z) = 2xy - y^2 + z$ at P = (0, 1, 1) in the direction of $v = 2i + j - k$. (2 pts)

$$\bar{u} = \frac{v}{|v|} = \frac{2i+j-k}{\sqrt{4+1+1}} = \frac{2}{\sqrt6} i + \frac{1}{\sqrt6} j - \frac{1}{\sqrt6} k$$

$$\nabla f(P) = \left[\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k\right](P) = \left[2yi + (2x-2y)j + k\right](0,1,1) = 2i - 2j + k$$

So, $D_{\bar u} f(P) = \nabla f(P)\cdot \bar u$

$$= (2i-2j+k)\cdot\left(\frac{2}{\sqrt6} i + \frac{1}{\sqrt6} j - \frac{1}{\sqrt6} k\right)$$

$$= \frac{4}{\sqrt6} - \frac{2}{\sqrt6} - \frac{1}{\sqrt6} = \boxed{\frac{1}{\sqrt6}}$$

6. Find the equations for the <u>tangent plane</u> and <u>normal line</u> for $x^3 - xy - y^2 - xz = 0$ at P = (1, 2, -1). (2 pts)

$$\nabla f(P) = \left[\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k\right](P) = \left[(3x^2-y-z)i + (-x-2y)j - xk\right](P) = 2i - 5j - k$$

Plane: $\frac{\partial f}{\partial x}(P)(x-x_0) + \frac{\partial f}{\partial y}(P)(y-y_0) + \frac{\partial f}{\partial z}(P)(z-z_0) = 0$

$$2(x-1) + (-5)(y-2) + (-1)(z+1) = 0$$

$$\boxed{2x - 5y - z = -7}$$

Line:
$$\left.\begin{aligned} x &= x_0 + \frac{\partial f}{\partial x}(P)t \\ y &= y_0 + \frac{\partial f}{\partial y}(P)t \\ z &= z_0 + \frac{\partial f}{\partial z}(P)t \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} x &= 1 + 2t \\ y &= 2 - 5t \\ z &= -1 - t \end{aligned}}$$

7. Given $w = xy - z$, where $x = \cos t,\ y = \sin t,\ z = t$, find $\frac{dw}{dt}$ at $t = \frac{\pi}{4}$. (3 pts)

Using Chain Rule : $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$

$= y(-\sin t) + x(\cos t) + (-1)\cdot 1$

$= \sin t(-\sin t) + \cos t(\cos t) - 1$

$= -\sin^2 t + \cos^2 t - 1$

$= \cos 2t - 1$

For $t = \frac{\pi}{4}$: $\left.\frac{dw}{dt}\right|_{t=\pi/4} = \cos 2(\pi/4) - 1 = \underbrace{\cancel{\cos\frac{\pi}{2}}}_{0} - 1 = \boxed{-1}$

8. Find all the local maxima, local minima, and saddle points of the following function:

$$f(x,y) = 4xy - x^4 - y^4 \qquad \text{(3 pts)}$$

$f_x = \frac{\partial f}{\partial x} = 4y - 4x^3$, $f_y = \frac{\partial f}{\partial y} = 4x - 4y^3$

For max, min, or saddle pts : $f_x = 0$ and $f_y = 0$.

ie, $\left.\begin{array}{l} 4y - 4x^3 = 0 \\ 4x - 4y^3 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y - x^3 = 0 \\ x - y^3 = 0 \end{array}\right\} \Rightarrow y - y^9 = 0$

$\Rightarrow y(1 - y^8) = 0$

$\Rightarrow y = 0$ or $y = \pm 1$

For $y = 0 \Rightarrow x = 0 \longrightarrow (0,0)$

For $y = 1 \Rightarrow x = 1 \longrightarrow (1,1)$

For $y = -1 \Rightarrow x = -1 \longrightarrow (-1,-1)$

So, a max, min or a saddle pt occur at $(1,1)$, $(-1,-1)$ or $(0,0)$.

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = -12x^2 \longrightarrow f_{xx}(1,1) = -12$, $f_{xx}(-1,-1) = -12$, $f_{xx}(0,0) = 0$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = -12y^2 \longrightarrow f_{yy}(1,1) = -12$, $f_{yy}(-1,-1) = -12$, $f_{yy}(0,0) = 0$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 4 \longrightarrow f_{xy}(1,1) = 4$, $f_{xy}(-1,-1) = 4$, $f_{xy}(0,0) = 4$

Since $f_{xx}(1,1) = -12 < 0$ and $H = f_{xx}(1,1) f_{yy}(1,1) - [f_{xy}(1,1)]^2 = 128 > 0 \longrightarrow (1,1)$ max.

Since $f_{xx}(-1,-1) = -12 < 0$ and $H = f_{xx}(-1,-1) f_{yy}(-1,-1) - [f_{xy}(-1,-1)]^2 = 128 > 0 \longrightarrow (-1,-1)$ max.

Finally, $H = f_{xx}(0,0) f_{yy}(0,0) - [f_{xy}(0,0)]^2 = -16 < 0 \longrightarrow (0,0)$ saddle.