Key

Midterm Exam

Math 200-Section B
(Fall 2005)

Solve the following problems. Show all your work in the space under each problem.

1. Find the equation of the plane that passes through the point P = (-1, 2, 1) and is perpendicular to the line of intersection of the planes $x-y+2z=3$ and $x+y-z=2$. (2 pts)

The vector $\bar{u}=\bar{n}_1\times\bar{n}_2$, where $\bar{n}_1=(1,-1,2)$ and $\bar{n}_2=(1,1,-1)$ is // to the line.

$$\bar{u}=\begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{vmatrix} = -i+3j+2k$$

The eqn of the plane is : $(-1)(x+1)+3(y-2)+2(z-1)=0$

$$\boxed{-x+3y+2z=9}$$

2. Find the length of the curve $\mathbf{r}(t)=(4t)i+(4t)j+(2-t)k$ from (0, 0, 2) to (4, 4, 1).
(*Note*: Make sure you write the correct limits during the calculation of the integral) (2 pts)

$0i+0j+2k=(4t)i+(4t)j+(2-t)k \Rightarrow t=0$
$4i+4j+k=(4t)i+(4t)j+(2-t)k \Rightarrow t=1$

$$S=\int_0^1 |r'(t)|\,dt=\int_0^1 |4i+4j-k|\,dt=\int_0^1 \sqrt{16+16+1}\,dt=\int_0^1\sqrt{33}\,dt$$
$$=[\sqrt{33}\,t]_0^1$$
$$=\boxed{\sqrt{33}}$$

3. Given $\mathbf{r}(t)=(3\sin t)i+(3\cos t)j+(4t)k$, show that $4\kappa+3\tau=0$, where κ and τ are the curvature and torsion, respectively, of the curve $\mathbf{r}(t)$. (2 pts)
(*Hint*: Use the formulas $\kappa=\dfrac{|v\times a|}{|v|^3}$ and $\tau=\dfrac{\det(\dot{x},\dot{y},\dot{z};\ddot{x},\ddot{y},\ddot{z};\dddot{x},\dddot{y},\dddot{z})}{|v\times a|^2}$)

$V=3\cos t\,i-3\sin t\,j+4k$, $a=-3\sin t\,i-3\cos t\,j+0k$

$$V\times a=\begin{vmatrix} i & j & k \\ 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \end{vmatrix} = 12\cos t\,i-12\sin t\,j+(-9\cos^2 t-9\sin^2 t)k$$
$$=12\cos t\,i-12\sin t\,j-9k$$

$$|V\times a|=\sqrt{144\cos^2 t+144\sin^2 t+81}=\sqrt{225}=15$$
$$|V|=\sqrt{9\cos^2 t+9\sin^2 t+16}=\sqrt{9+16}=\sqrt{25}=5$$

So, $\kappa=\dfrac{15}{5^3}=\dfrac{3\cdot\cancel{5}}{\cancel{5}\cdot 5^2}=\dfrac{3}{25}$

Now, $\det(\dot{x},\dot{y},\dot{z};\ddot{x},\ddot{y},\ddot{z};\dddot{x},\dddot{y},\dddot{z})=\begin{vmatrix} 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \\ -3\cos t & 3\sin t & 0 \end{vmatrix}=4(-9\sin^2 t-9\cos^2 t)$
$=4(-9)$
$=-36$

So, $\tau=\dfrac{-36}{15^2}=\dfrac{-4\cdot\cancel{9}}{(3\cdot5)(3\cdot5)}=\dfrac{-4}{25}$

Therefore, $4\kappa+3\tau=4(3/25)+3(-4/25)=12/25-12/25=0$.

4. (a) Calculate the limit: $\lim_{(x,y)\to(\frac{\pi}{2},0)} \dfrac{2-xy+2\sin x}{\cos y+1}$ (4 pts)

$$\lim_{(x,y)\to(\pi/2,0)} \frac{2-xy+2\sin x}{\cos y+1} = \frac{2-\pi/2\cdot 0+2\sin \pi/2}{\cos 0+1} = \frac{2-0+2}{1+1} = \frac{4}{2} = \boxed{2}$$

(b) Let $f(x)=\dfrac{x^3-xy^2}{x^3+y^3}$. Show that $f(x)$ has no limit as $(x,y)\to(0,0)$.

Consider the paths : $y=kx$, $k\in\mathbb{R}$.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} \frac{x^3-xy^2}{x^3+y^3} = \lim_{(x,y)\to(0,0)} \frac{x^3-x(k^2x^2)}{x^3+k^3x^3} = \lim_{(x,y)\to(0,0)} \frac{\cancel{x^3}(1-k^2)}{\cancel{x^3}(1+k^3)} = \lim_{(x,y)\to(0,0)} \frac{1-k^2}{1+k^3} = \frac{1-k^2}{1+k^3}$$

Hence, for different k we get different values for the limit, ie f has <u>no</u> limit .

5. Find the directional derivative of $f(x,y,z)=2xy-x^2+z$ at P = (1, 0, 1) in the direction of $v=i-j+k$. (2 pts)

$$\bar{u} = \frac{v}{|v|} = \frac{i-j+k}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}i - \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k$$

$$\nabla f(P) = \left[\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k\right](P) = \left[(2y-2x)i+2xj+k\right](1,0,1)$$
$$= -2i+2j+k$$

$$\text{So, } D_{\bar{u}}f(P) = \nabla f(P)\cdot\bar{u}$$
$$= (-2i+2j+k)\cdot(1/\sqrt{3}\,i - 1/\sqrt{3}\,j + 1/\sqrt{3}\,k)$$
$$= -2/\sqrt{3} - 2/\sqrt{3} + 1/\sqrt{3} = \boxed{-3/\sqrt{3}}$$

6. Find the equations for the <u>tangent plane</u> and <u>normal line</u> for $y^3-xy-x^2-yz=0$ at P = (1, -1, 0). (2 pts)

$$\nabla f(P) = \left[\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k\right](P) = \left[(-y-2x)i+(3y^2-x-z)j-yk\right](P)$$
$$= -i+2j+k$$

Plane : $\frac{\partial f}{\partial x}(P)(x-x_0) + \frac{\partial f}{\partial y}(P)(y-y_0) + \frac{\partial f}{\partial z}(P)(z-z_0) = 0$

$$(-1)(x-1)+2(y+1)+1(z-0)=0$$

$$\boxed{-x+2y+z=-3}$$

Line :
$$\left.\begin{aligned} x &= x_0 + \frac{\partial f}{\partial x}(P)t \\ y &= y_0 + \frac{\partial f}{\partial y}(P)t \\ z &= z_0 + \frac{\partial f}{\partial z}(P)t \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} x &= 1-t \\ y &= -1+2t \\ z &= t \end{aligned}}$$

7. Given $w = xy - z$, where $x = \sin t$, $y = 2\cos t$, $z = t$, find $\frac{dw}{dt}$ at $t = \frac{\pi}{2}$. (3 pts)

Using Chain Rule : $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$

$$= y\cos t + x(-2\sin t) + (-1)\cdot 1$$
$$= 2\cos t\cos t + \sin t(-2\sin t) - 1$$
$$= 2\cos^2 t - 2\sin^2 t - 1$$
$$= 2\cos 2t - 1$$

For $t = \pi/2$: $\left.\frac{dw}{dt}\right|_{t=\pi/2} = 2\cos 2(\pi/2) - 1 = 2\overset{-1}{\cos\pi} - 1$
$$= -2 - 1 = \boxed{-3}$$

8. Find all the local maxima, local minima, and saddle points of the following function:

$$f(x, y) = 4xy - x^4 - y^4 \qquad \text{(3 pts)}$$

$f_x = \frac{\partial f}{\partial x} = 4y - 4x^3$, $f_y = \frac{\partial f}{\partial y} = 4x - 4y^3$

For max, min or saddle pts : $f_x = 0$ and $f_y = 0$

ie, $\left.\begin{matrix} 4y - 4x^3 = 0 \\ 4x - 4y^3 = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} y - x^3 = 0 \\ x - y^3 = 0 \end{matrix}\right\} \Rightarrow y - y^9 = 0 \Rightarrow y(1 - y^8) = 0$
$$\Rightarrow y = 0 \text{ or } y = \pm 1$$

For $y = 0 \Longrightarrow x = 0 \longrightarrow (0,0)$
For $y = 1 \Longrightarrow x = 1 \longrightarrow (1,1)$
For $y = -1 \Longrightarrow x = -1 \longrightarrow (-1,-1)$

So, a max, min or saddle pt occur at $(0,0)$, $(1,1)$ or $(-1,-1)$.

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = -12x^2 \longrightarrow f_{xx}(1,1) = -12$, $f_{xx}(-1,-1) = -12$, $f_{xx}(0,0) = 0$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = -12y^2 \longrightarrow f_{yy}(1,1) = -12$, $f_{yy}(-1,-1) = -12$, $f_{yy}(0,0) = 0$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 4 \longrightarrow f_{xy}(1,1) = 4$, $f_{xy}(-1,-1) = 4$, $f_{xy}(0,0) = 4$

Since $f_{xx}(1,1) = -12 < 0$ and $H = f_{xx}(1,1)f_y(1,1) - [f_{xy}(1,1)]^2 = 128 > 0 \longrightarrow (1,1)$ max

Since $f_{xx}(-1,-1) = -12 < 0$ and $H = f_{xx}(-1,-1)f_y(-1,-1) - [f_{xy}(-1,-1)]^2 = 128 > 0 \longrightarrow (-1,-1)$ max

Finally, $H = f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2 = -16 < 0 \longrightarrow (0,0)$ saddle .