Math 200 A

Homework # 1

1. If $\vec{u} = 2i - k$, let $\vec{v} = -\vec{u} = -2i + k$. $\vec{v}$ is in opposite direction as $\vec{u}$.

$|\vec{v}| = \sqrt{4+1} = \sqrt{5} \rightarrow$ scale $\vec{v}$ by $\frac{3}{\sqrt{5}} \rightarrow \boxed{\vec{v} = \frac{-6}{\sqrt{5}} i + \frac{3}{\sqrt{5}} k}$

now $|\vec{v}| = \sqrt{\frac{36}{5} + \frac{9}{5}} = \sqrt{\frac{45}{5}} = 3$

2. $3x^2 + 3y^2 + 3z^2 + 2y - 2z = 9 \rightarrow x^2 + y^2 + z^2 + \frac{2}{3}y - \frac{2}{3}z = 3$

$x^2 + y^2 + \frac{2}{3}y + z^2 - \frac{2}{3}z = 3$

$x^2 + y^2 + \frac{2}{3}y + \frac{1}{9} + z^2 - \frac{2}{3}z + \frac{1}{9} = 3 + \frac{1}{9} + \frac{1}{9}$

$x + (y + \frac{1}{3})^2 + (z - \frac{1}{3})^2 = 3 + \frac{2}{9}$

$\boxed{\text{center} = (0, -\frac{1}{3}, \frac{1}{3})}$ radius $= \sqrt{3 + \frac{2}{9}} = \sqrt{\frac{29}{9}} = \boxed{\frac{\sqrt{29}}{3}}$

3. $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$

$u = i \times j = \det \begin{bmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = (0-0)i - (0-0)j + (1-0)k$

$u = (0,0,1)$

$v = j \times k = \det \begin{bmatrix} i & j & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1-0)i - (0-0)j + (0-0)k$

$v = (1,0,0)$

$v \times u = \det \begin{bmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (0-0)i - (1-0)j + (0-0)k$

$v \times u = -j \rightarrow v \times u = (0,-1,0)$

4. volume $= (u \times v) \cdot w = \det \begin{bmatrix} i & j & k \\ 1 & 1 & -2 \\ -1 & 0 & -1 \end{bmatrix} \cdot w = ((-1-0)i - (-1-2)j + (0+1)k) \cdot w$

$= (-1, 3, 1) \cdot (2, 4, -2) = -2 + 12 - 2 = \boxed{8}$

5. x-intercept when $t=2$; $y=3$ when $t=2$, $z=-1$ when $t=2$
line meets the $x=0$ plane at $(0,3,-1)$
y-intercept when $t=1$, $x=3$, $z=-4$
line meets the $y=0$ plane at $(3,0,-4)$.
z-intercept when $t=3$, $x=-1$, $y=4$
line meets the $z=0$ plane at $(-1,4,0)$.

6. Given that $r'(t)$ exists, must show that
$\lim_{t\to c}(r(t)) = r(c) \Rightarrow \lim_{h\to 0}(r(c+h)) = r(c)$

$r(c+h) = r(c) + (r(c+h) - r(c))$
$r(c+h) = r(c) + \frac{r(c+h)-r(c)}{h} h$
$\lim_{h\to 0}(r(c+h)) = \lim_{h\to 0}(r(c)) + \lim_{h\to 0}\left(\frac{r(c+h)+r(c)}{h}\right)\cdot \lim_{h\to 0}(h)$
$\lim_{h\to 0}(r(c+h)) = r(c) + r'(c)(0)$
$\lim_{h\to 0}(r(c+h)) = r(c) \rightarrow$ let $c=t_0$; $\therefore$ if $r'(t_0)\ \exists$, then $r(t_0)$ is continuous.
However, the converse is not true. Consider $f(t)=|t|$, $f'(t)\ \nexists$ if $t=0$.
$\therefore f(t)$ is not differentiable at $t=0$ but is continuous at $t=0$. $r(t)$ fails to be differentiable or continuous where $f(t)$ (or $g(t)$ or $h(t)$) also fails to

7. $\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = (v_2w_3 - w_2v_3)u_1 - (v_1w_3 - w_1v_3)u_2 + (v_1w_2 - w_1v_2)u_3$

$$\frac{d}{dt}(\cdot) = \begin{cases} u_1\left(v_2\frac{dw_3}{dt} + w_3\frac{dv_2}{dt}\right) + \frac{du_1}{dt}(v_2w_3) - u_1\left(w_2\frac{dv_3}{dt} + v_3\frac{dw_2}{dt}\right) - \frac{du_1}{dt}(w_2v_3) \\ + u_2\left(w_1\frac{dv_3}{dt} + v_3\frac{dw_1}{dt}\right) + \frac{du_2}{dt}(w_1v_3) - u_2\left(v_1\frac{dw_3}{dt} + w_3\frac{dv_1}{dt}\right) - \frac{du_2}{dt}(v_1w_3) \\ + u_3\left(v_1\frac{dw_2}{dt} + w_2\frac{dv_1}{dt}\right) + \frac{du_3}{dt}(v_1w_2) - u_3\left(w_1\frac{dv_2}{dt} + v_2\frac{dw_1}{dt}\right) - \frac{du_3}{dt}(w_1v_2) \end{cases}$$

$$= (v_2w_3 - w_2v_3)\frac{du_1}{dt} - (v_1w_3 - w_1v_3)\frac{du_2}{dt} + (v_1w_2 - w_1v_2)\frac{du_3}{dt} + [\text{terms without } \tfrac{du}{dt}]$$

$$= \det\begin{bmatrix} \frac{du_1}{dt} & \frac{du_2}{dt} & \frac{du_3}{dt} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} + \det\begin{bmatrix} u_1 & u_2 & u_3 \\ \frac{dv_1}{dt} & \frac{dv_2}{dt} & \frac{dv_3}{dt} \\ w_1 & w_2 & w_3 \end{bmatrix} + \det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \frac{dw_1}{dt} & \frac{dw_2}{dt} & \frac{dw_3}{dt} \end{bmatrix}$$

same as first one

8. $r'(t) = v(t) = 3i + 3j - k$, start $= (0,0,2)$, end $(3,3,1)$

$\therefore$ need to move by the vector $(3-0, 3-0, 1-2) = (3,3,-1) = v$

clearly, the start point is when $t=0$ & endpoint is when $t=1$

$s = \int_0^1 |v|\,dt = \int_0^1 \sqrt{9+9+1}\,dt = \int_0^1 (\sqrt{19})\,dt \to t\sqrt{19}\Big|_0^1 = \boxed{\sqrt{19}}$

9. $r'(t) = v(t) = i + (\cos t)j \quad |v| = \sqrt{1+\cos^2 t}$

$T = \frac{v}{|v|} = (1+\cos^2 t)^{-1/2}\, i + (\cos t)(1+\cos^2 t)^{-\frac{1}{2}}$

$\frac{dT}{dt} = -\frac{1}{2}(1+\cos^2 t)^{-\frac{3}{2}}(2\cos t)(-\sin t)i + (\cos t)(-\frac{1}{2}(1+\cos^2 t)^{-\frac{3}{2}}(2\cos t)(-\sin t)\,j -$

$= (1+\cos^2 t)^{-\frac{3}{2}}(\cos t)(\sin t)i + \left((1+\cos^2 t)^{-\frac{3}{2}}(\cos^2 t)(\sin t) - (\sin t)(1+\cos^2 t)^{-\frac{1}{2}}\right)j \quad (\sin t)(1+\cos^2 t)^{-\frac{1}{2}}j$

$= (1+0)^{-\frac{3}{2}}(0)(1)\,i + \left((1+0)^{-\frac{3}{2}}(0)(1) - (1)(1+0)^{-\frac{1}{2}}\right)j = 0i - j$

$\frac{dT}{dt} = -j$ when $t = \frac{5\pi}{2}$, $K = \frac{1}{|v|}\left|\frac{dT}{dt}\right| \to K = \frac{1}{|v|} = \frac{1}{\sqrt{1+\cos^2 t}} = \frac{1}{\sqrt{1}} \to \boxed{K=1}$

when $t = \frac{5\pi}{2}$, $T = (1+\cos^2(\frac{5\pi}{2}))^{-\frac{1}{2}}\, i + (\cos^2(\frac{5\pi}{2}))^{-\frac{1}{2}} j \to T = i + 0j = i$

center is at $(\frac{5\pi}{2}, 0)$

$\therefore$ equation of the circle of curvature: $(x - \frac{5\pi}{2})^2 + (y-0)^2 = 1$

10. $r(t) = \left(\frac{t^3}{3}\right)i + \left(\frac{t^2}{2}\right)j$

$r'(t) = v(t) = t^2 i + t j \quad |v| = \sqrt{t^4+t^2}$

$T = \frac{v}{|v|} = (t^4+t^2)^{-\frac{1}{2}}(t^2)i + (t^4+t^2)^{-\frac{1}{2}}(t)\,j$

$\frac{dT}{dt} = \left[(t^4+t^2)^{-\frac{1}{2}}(2t) + (t^2)(-\frac{1}{2})(t^4+t^2)^{-\frac{3}{2}}(4t^3+2t)\right]i + \left[(t^4+t^2)^{-\frac{1}{2}} + (t)(-\frac{1}{2})(t^4+t^2)^{-\frac{3}{2}}(4t^3+2t)\right]j$

$K = \frac{1}{|v|}\left|\frac{dT}{dt}\right|$

$K = \sqrt{\left[(t^4+t^2)^{-1}(2t) - (t^2)(t^4+t^2)^{-2}(2t^3+t)\right]^2 + \left[(t^4+t^2)^{-1} - (t)(t^4+t^2)^{-2}(2t^3+t)\right]^2}$

$N = \frac{dT}{dt} \Big/ |dT/dt| \qquad B = T \times N$

$r''(t) = a(t) = 2t + j$

$$\tau = \frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{|v \times a|^2} = \frac{\begin{vmatrix} t^2 & t & 0 \\ 2t & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix}}{|v \times a|^2} = t^2(0-0) - t(0-0) + 0(0-2) = \frac{0}{|v \times a|^2} = \boxed{0}$$