Homework 1 Math 200 - Section B Key

1. A vector in the direction opposite of $\vec{u} = \hat{\imath} - 2\hat{k}$ is $-\hat{\imath} + 2\hat{k}$.

A unit vector in this direction is

$$\frac{-\hat{\imath} + 2\hat{k}}{\sqrt{(-1)^2 + 2^2}} = -\frac{1}{\sqrt{5}}\hat{\imath} + \frac{2}{\sqrt{5}}\hat{k}.$$

So a vector of magnitud 2 in this direction is $2\left(-\frac{1}{\sqrt{5}}\hat{\imath} + \frac{2}{\sqrt{5}}\hat{k}\right) = \boxed{-\frac{2}{\sqrt{5}}\hat{\imath} + \frac{4}{\sqrt{5}}\hat{k}}$

2. $$2x^2 + 2y^2 + 2z^2 + x - y + 2z = 5$$

$$\Rightarrow x^2 + \tfrac{1}{2}x + y^2 - \tfrac{1}{2}y + z^2 + z = \tfrac{5}{2}$$

$$\Rightarrow x^2 + \tfrac{1}{2}x + \tfrac{1}{16} + y^2 - \tfrac{1}{2}y + \tfrac{1}{16} + z^2 + z + \tfrac{1}{4} = \tfrac{5}{2} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{4}$$

$$\Rightarrow (x + \tfrac{1}{4})^2 + (y - \tfrac{1}{4})^2 + (z + \tfrac{1}{2})^2 = \tfrac{23}{8}$$

This is a circle of radius $\sqrt{\frac{23}{8}}$ centered at $\left(-\frac{1}{4}, \frac{1}{4}, -\frac{1}{2}\right)$

3. $\vec{u} = \hat{\imath} \times \hat{\jmath} = \hat{k}$

$\vec{v} = \hat{\jmath} \times \hat{k} = \hat{\imath}$

So $\vec{v} \times \vec{u} = \hat{\imath} \times \hat{k} = -\hat{\jmath}$, length $= 1$, direction $= -\hat{\jmath}$

4. $$\begin{aligned} \text{volume} &= |(\vec{u} \times \vec{v}) \cdot \vec{w}| \\ &= |(\hat{\imath} - 2\hat{\jmath} - 4\hat{k}) \cdot (\hat{\imath} + 2\hat{k})| \\ &= |1 + 0 - 8| \\ &= \boxed{7} \end{aligned}$$

5. $\begin{cases} x = 3-t \\ y = 2t-1 \\ z = 2t-2 \end{cases}$

The line meets the xy-plane when $z=0$, i.e. $2t-2=0$, ie. when $t=1$. So at the point $(3-(1), 2(1)-1, 0) = (2, 1, 0)$.

It meets the xz-plane when $y=0$, ie. $2t-1=0$, ie. when $t=\frac{1}{2}$. So at the point $(3-(\frac{1}{2}), 0, 2(\frac{1}{2})-2) = (\frac{5}{2}, 0, -1)$

It meets the yz-plane when $x=0$, ie. $3-t=0$, ie. when $t=3$. So at the point $(0, 2(3)-1, 2(3)-2) = (0, 5, 4)$

6. Since $\vec{r}$ is differentiable at t_0, we know that $\vec{r}'$ is defined at $t=t_0$, i.e., $r'(t_0)$ exists. Since $r'(t_0)$ exists, and $r'(t) = f'(t)\hat{i} + g'(t)\hat{j} + h'(t)\hat{k}$, then $f'(t_0)\hat{i} + g'(t_0)\hat{j} + h'(t_0)\hat{k}$ exists, so $f'(t_0)$, $g'(t_0)$, and $h'(t_0)$ all exist. Since f', g', and h' are all defined ~~at~~ at $t=t_0$, f, g, and h are all continuous at $t=t_0$, there fore $\vec{r}(t)$ is continuous at $t=t_0$.

The convers is not true. Example: let $\vec{r} = |t|\hat{i}$, then $\vec{r}$ is continuous but not differentiable at $t=0$.

7. By using the formula for the Triple Scalar Product (Section 10.2, page 804 in the textbook), we can rewrite the equation as:

$$\frac{d}{dt}(\vec{u}\cdot\vec{v}\times\vec{w}) = \frac{d\vec{u}}{dt}\cdot\vec{v}\times\vec{w} + \vec{u}\cdot\frac{d\vec{v}}{dt}\times\vec{w} + \vec{u}\cdot\vec{v}\times\frac{d\vec{w}}{dt}$$

We must show this is true.

Looking at the left hand side of the equation, we get:

$$\frac{d}{dt}(\vec{u}\cdot\vec{v}\times\vec{w}) = \frac{d\vec{u}}{dt}\cdot(\vec{v}\times\vec{w}) + \vec{u}\cdot\frac{d}{dt}(\vec{v}\times\vec{w})$$

$$= \frac{d\vec{u}}{dt}\cdot\vec{v}\times\vec{w} + \vec{u}\cdot\left(\frac{d\vec{v}}{dt}\times\vec{w} + \vec{v}\times\frac{d\vec{w}}{dt}\right)$$

$$= \frac{d\vec{u}}{dt}\cdot\vec{v}\times\vec{w} + \vec{u}\cdot\frac{d\vec{v}}{dt}\times\vec{w} + \vec{u}\cdot\vec{v}\times\frac{d\vec{w}}{dt}.$$

Done.

8. $\vec{r} = 2t\hat{i} + 2t\hat{j} + (1-t)\hat{k}$

$\vec{v} = \frac{d\vec{r}}{dt} = 2\hat{i} + 2\hat{j} - \hat{k}$

$|\vec{v}| = \sqrt{4+4+1} = 3$

$\vec{r} = (0,0,1)$ when $t=0$

$\vec{r} = (2,2,0)$ when $t=1$

length is then $\int_{t_i}^{t_f} |v|\,dt = \int_0^1 3\,dt = 3t\Big|_0^1 = 3-0 = \boxed{3}$

9. $\vec{r} = t\hat{\imath} + (\sin(t))\hat{\jmath}$ $\qquad\qquad$ $\vec{r} = (\frac{\pi}{2}, 1)$ when $t = \frac{\pi}{2}$

$\vec{v} = \frac{d\vec{r}}{dt} = \hat{\imath} + (\cos(t))\hat{\jmath}$

$|\vec{v}| = \sqrt{1+\cos^2(t)}$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{1+\cos^2(t)}}\hat{\imath} + \frac{\cos(t)}{\sqrt{1+\cos^2(t)}}\hat{\jmath}$$

$$\frac{d\vec{T}}{dt} = \sin(t)\cos(t)\left(1+\cos^2(t)\right)^{-3/2}\hat{\imath} + \Big[\sin(t)\cos^2(t)\left(1-\cos^2(t)\right)^{-3/2} - \cdots$$
$$\hookrightarrow -\sin(t)\left(1+\cos^2(t)\right)^{-1/2}\Big]\hat{\jmath}$$

$K = \frac{1}{|\vec{v}|}\left|\frac{dT}{dt}\right|$. At $t = \frac{\pi}{2}$, this is $\frac{1}{\sqrt{1}}\left|0\hat{\imath} - \hat{\jmath}\right| = (1)\sqrt{0+1} = 1$

So $\rho = \frac{1}{K} = 1$

Since $\frac{dT}{dt} = -\hat{\jmath}$ at $t = \frac{\pi}{2}$ the curve is concave down at $(\frac{\pi}{2}, 1)$ and the center of the circle is at $(\frac{\pi}{2}, 0)$.

So the equation for the circle of curvature is $\boxed{(x-\frac{\pi}{2})^2 + (y-0)^2 = 1}$

10. $\vec{r} = \frac{t^3}{3}\hat{\imath} + \frac{t^2}{2}\hat{\jmath}$, $t > 0$

$\vec{v} = \frac{d\vec{r}}{dt} = t^2\hat{\imath} + t\hat{\jmath}$, $t > 0$ $\qquad |\vec{v}| = \sqrt{t^4 + t^2} = t\sqrt{t^2+1}$, since $t > 0$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \boxed{\frac{t}{\sqrt{t^2+1}}\hat{\imath} + \frac{1}{\sqrt{t^2+1}}\hat{\jmath} = \vec{T}}$$

$$\frac{d\vec{T}}{dt} = (t^2+1)^{-3/2}\hat{\imath} - t(t^2+1)^{-3/2}\hat{\jmath}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{(t^2+1)^{-3} + t^2(t^2+1)^{-3}} = \sqrt{(t^2+1)^{-3}(1+t^2)} = \sqrt{(t^2+1)^{-2}} = \frac{1}{t^2+1}$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = \frac{t^2+1}{(t^2+1)^{3/2}}\hat{\imath} - \frac{t(t^2+1)}{(t^2+1)^{3/2}}\hat{\jmath} = \boxed{\frac{1}{\sqrt{t^2+1}}\hat{\imath} - \frac{t}{\sqrt{t^2+1}}\hat{\jmath} = \vec{N}}$$

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{t}{\sqrt{t^2+1}} & \frac{1}{\sqrt{t^2+1}} & 0 \\ \frac{1}{\sqrt{t^2+1}} & \frac{-t}{\sqrt{t^2+1}} & 0 \end{vmatrix} = \frac{-t^2-1}{t^2+1}\hat{k} = \boxed{-\hat{k} = \vec{B}}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = 2t\hat{\imath} + \hat{\jmath}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ t^2 & t & 0 \\ 2t & 1 & 0 \end{vmatrix} = -t^2 k \qquad |\vec{v} \times \vec{a}| = t^2$$

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{t^2}{(t\sqrt{t^2+1})^3} = \boxed{\frac{1}{t(t^2+1)^{3/2}} = \kappa}$$

$$\frac{d\vec{a}}{dt} = 2\hat{\imath}$$

$$\tau = \frac{\begin{vmatrix} t^2 & t & 0 \\ 2t & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix}}{|\vec{v} \times \vec{a}|^2} = \boxed{0 = \tau}$$