QUIZ 2

(Math 200-Section A)

1. Find the vectors **T**, **N** and **B** for the curve $\mathbf{r}(t) = ti + \frac{1}{2}e^{2t} j$. (4 pts)

$\frac{dr}{dt} = \hat{i} + e^{2t}\hat{j}$ $\quad \vec{T} = \frac{\frac{dr}{dt}}{\left|\frac{dr}{dt}\right|}$ $\quad \vec{T} = \frac{1}{\sqrt{1+e^{4t}}}\hat{i} + \frac{e^{2t}}{\sqrt{1+e^{4t}}}\hat{j}$

$\left|\frac{dr}{dt}\right| = \sqrt{1+e^{4t}}$

$\vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$ $\quad \vec{B} = \vec{T} \times \vec{N}$

2. Find the curvature κ and the torsion τ, at the given value of t $(t=0)$, for the curve $\mathbf{r}(t) = (e^t \sin 2t)i + (e^t \cos 2t)j + 2e^t k$. (4 pts)

$r(t) = (e^t \sin 2t)\hat{i} + (e^t \cos 2t)\hat{j} + 2e^t k.$

$v(t) = \frac{dr}{dt} = (e^t 2\cos 2t + e^t \sin 2t)\hat{i} + (-e^t 2\sin 2t + e^t \cos 2t)\hat{j} + (2e^t)k$

$a(t) = \frac{d^2r}{dt^2} = (-e^t 4\sin 2t + e^t 2\cos 2t + e^t 2\cos 2t + e^t \sin 2t)\hat{i} + (-e^t 4\cos 2t + -e^t 2\sin 2t + e^t 2\sin 2t + e^t \cos 2t)\hat{j} + (2e^t)k$

$a(t) = e^t(-3\sin 2t + 4\cos 2t)\hat{i} + e^t(-3\cos 2t - 4\sin 2t)\hat{j} + (2e^t)k$

$k = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$ $\quad \tau = \frac{r' \cdot (r' \times r'')}{|\vec{v} \times a|^2}$

3. Given $\mathbf{r}(t) = (2+t)i + (t+2t^2)j + (1+t^2)k$, write the acceleration **a** in the form **a** = a_T**T** + a_N**N** at the given value of t $(t=0)$, without finding the vectors **T** and **N**. (4 pts)

$v(t) = 1\,i + (1+4t)j + 2t\,k$

$|v| = \sqrt{1 + 1 + 8t + 16t^2 + 4t^2} = \sqrt{2+8t+20t^2} = (2+8t+20t^2)^{\frac{1}{2}}$

$a_T = \frac{d}{dt}|v| = \frac{8+40t}{2\sqrt{2+8t+20t^2}} = \frac{8}{2\sqrt{2}}$

$a(t) = 4j + 2k$

$|a|^2 = 16 + 4 = 20$

$a_N = \sqrt{|a|^2 - a_T^2} = \sqrt{20 - a_T^2} = \sqrt{20 - \left(\frac{8}{2\sqrt{2}}\right)^2}$

$\boxed{\vec{a}(0) = \frac{8}{2\sqrt{2}}\vec{T} + \sqrt{20 - \left(\frac{8}{2\sqrt{2}}\right)^2}\,\vec{N}}$

4. Show that the curvature κ of the line $\mathbf{r}(t) = (x_0 + At)i + (y_0 + Bt)j + (z_0 + Ct)k$ is zero. What is its torsion τ ? (4 pts)

Cannot use $\tau = \frac{r' \cdot (r'' \times r''')}{|v \times a|^2}$ to find τ, because this formula applies only for $v \times a \neq 0$.

But since the line $r(t)$ is a planar curve $\Rightarrow$ $\tau = 0$

5. Show that the curvature κ of a smooth curve $\mathbf{r}(t) = f(t)i + g(t)j$ defined by the twice-differentiable functions $x = f(t)$ and $y = g(t)$ is given by the formula:

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$ (4 pts)

$k = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$

$|\vec{v} \times \vec{a}| = \begin{vmatrix} i & j & k \\ \dot{x} & \dot{y} & 0 \\ \ddot{x} & \ddot{y} & 0 \end{vmatrix} = 0i + 0j + (\dot{x}\ddot{y} - \ddot{x}\dot{y})k$

$|v|^3 = \left(\sqrt{\dot{x}^2 + \dot{y}^2}\right)^3 = (\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}$

$k = \frac{(\dot{x}\ddot{y} - \ddot{x}\dot{y})}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}$