Key

QUIZ 2

(Math 200-Section B)

1. Find the vectors **T**, **N** and **B** for the curve $\mathbf{r}(t) = (e^t \cos t)i + (e^t \sin t)j$. (4 pts)

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(e^t\cos(t) - e^t\sin(t)\right)\hat{i} + \left(e^t\sin(t) + e^t\cos(t)\right)\hat{j}$$

$$|\vec{v}| = \sqrt{\left(e^t\cos(t) - e^t\sin(t)\right)^2 + \left(e^t\sin(t) + e^t\cos(t)\right)^2}$$

$$= \sqrt{2}\cdot e^t$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\left(\cancel{e^t}\cos(t) - \cancel{e^t}\sin(t)\right)\hat{i} + \left(\cancel{e^t}\sin(t) + \cancel{e^t}\cos(t)\right)\hat{j}}{\sqrt{2}\,\cancel{e^t}}$$

$$= \boxed{\tfrac{1}{\sqrt{2}}\left(\cos(t) - \sin(t)\right)\hat{i} + \tfrac{1}{\sqrt{2}}\left(\sin(t) + \cos(t)\right)\hat{j}} = (T_i)\hat{i} + (T_j)\hat{j}$$

$$\frac{d}{dt}\vec{T} = \tfrac{1}{\sqrt{2}}\left(-\sin(t) - \cos(t)\right)\hat{i} + \tfrac{1}{\sqrt{2}}\left(\cos(t) - \sin(t)\right)\hat{j}$$

$$\left|\frac{d}{dt}\vec{T}\right| = \sqrt{\tfrac{1}{2}\left(-\sin(t) - \cos(t)\right)^2 + \tfrac{1}{2}\left(\cos(t) - \sin(t)\right)^2} = 1$$

$$\vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|} = \boxed{\tfrac{1}{\sqrt{2}}\left(-\sin(t) - \cos(t)\right)\hat{i} + \tfrac{1}{\sqrt{2}}\left(\cos(t) - \sin(t)\right)\hat{j}} = (N_i)\hat{i} + (N_j)\hat{j}$$

$$\vec{B} = \vec{T}\times\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ T_i & T_j & 0 \\ N_i & N_j & 0 \end{vmatrix} = \hat{k}\left(T_iN_j - T_jN_i\right) = \left(\tfrac{1}{2}\left(\cos(t) - \sin(t)\right)^2 + \tfrac{1}{2}\left(\cos(t) + \sin(t)\right)^2\right)\hat{k} = \boxed{\hat{k}}$$

2. Find the curvature κ and the torsion τ, at the given value of t $(t = 0)$, for the curve $\mathbf{r}(t) = (\cos^3 t)i + (\sin^3 t)j$. (4 pts)

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(3\cos^2(t)\right)\left(-\sin(t)\right)\hat{i} + \left(3\sin^2(t)\right)\left(\cos(t)\right)\hat{j} = (v_i)\hat{i} + (v_j)\hat{j} + 0\hat{k}$$

$$|\vec{v}| = \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)} = 3\sin(t)\cos(t)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(9\sin^2(t)\cos(t) - 3\cos(t)\right)\hat{i} + \left(6\cos^2(t)\sin(t) - 3\sin^3(t)\right)\hat{j}$$

$$\vec{v}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_i & v_j & 0 \\ a_i & a_j & 0 \end{vmatrix} = \hat{k}\left(v_ia_j - v_ja_i\right) = \left(-9\sin^2(t)\cos^2(t)\right)\hat{k}$$

$$|\vec{v}\times\vec{a}| = 9\cos^2(t)\sin^2(t)$$

$$\kappa = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3} = \frac{9\cos^2(t)\sin^2(t)}{27\sin^3(t)\cos^3(t)} = \boxed{\frac{1}{3\sin(t)\cos(t)}}$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v}\times\vec{a}|^2} = \frac{\begin{vmatrix} v_i & v_j & 0 \\ a_i & a_j & 0 \\ \dddot{x} & \dddot{y} & 0 \end{vmatrix}}{|\vec{v}\times\vec{a}|^2} = \frac{0}{|\vec{v}\times\vec{a}|^2} = \boxed{0}$$

3. Given $\mathbf{r}(t) = \frac{t^2}{2} i + (3-t) j + 2tk$, write the acceleration $\mathbf{a}$ in the form $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$ at the given value of t $(t = 1)$, without finding the vectors $\mathbf{T}$ and $\mathbf{N}$. (4 pts)

$\vec{v} = \frac{d\vec{r}}{dt} = t\hat{i} - \hat{j} + 2\hat{k}$ $\qquad$ $\vec{a} = \frac{d\vec{v}}{dt} = \hat{i}$ $\qquad$ $|\vec{a}| = 1$

$|\vec{v}| = \sqrt{t^2+1+4} = \sqrt{t^2+5}$

$a_T = \frac{d}{dt}|\vec{v}| = \frac{d}{dt}(t^2+5)^{1/2} = \cancel{\frac{1}{2}}(t^2+5)^{-1/2}\cdot(\cancel{2}t) = \frac{t}{\sqrt{t^2+5}}$ $\quad$ at $t=1$, this is $\frac{1}{\sqrt{6}}$

$a_N = \sqrt{|a|^2 - a_T^2}$ $\quad$ at $t=1$, this is $\sqrt{1-\frac{1}{6}}$

$= \sqrt{1 - \frac{t^2}{t^2+5}}$ $\qquad$ $= \sqrt{\frac{5}{6}}$

So $\vec{a} = a_T\vec{T} + a_N\vec{N} = \boxed{\frac{1}{\sqrt{6}}\vec{T} + \frac{\sqrt{5}}{\sqrt{6}}\vec{N}}$

4. Show that the curvature κ of the line $\mathbf{r}(t) = (x_0 + At)i + (y_0 + Bt)j + (z_0 + Ct)k$ is zero. What is its torsion τ ? (4 pts)

$\vec{V} = \frac{d\vec{r}}{dt} = A\hat{i} + B\hat{j} + C\hat{k}$

$\vec{a} = \frac{d\vec{v}}{dt} = \vec{0}$

$|\vec{v}\times\vec{a}| = |\vec{v}\times\vec{0}| = |\vec{0}| = 0$, so $\kappa = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3} = \frac{0}{|\vec{v}|^3} = 0$

$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v}\times\vec{a}|^2} = \frac{\begin{vmatrix} A & B & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}}{0^2} = \frac{0}{0}$, undefined, so we say $\tau = 0$

5. Show that the curvature κ of a smooth curve $\mathbf{r}(t) = f(t)i + g(t)j$ defined by the twice-differentiable functions $x = f(t)$ and $y = g(t)$ is given by the formula:

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$$

(4 pts)

We have $\vec{r} = x\hat{i} + y\hat{j}$,

so $\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$ and $\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$.

Now, $\vec{v}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x} & \dot{y} & 0 \\ \ddot{x} & \ddot{y} & 0 \end{vmatrix} = \hat{k}(\dot{x}\ddot{y} - \dot{y}\ddot{x})$ thus $|\vec{v}\times\vec{a}| = \sqrt{(\dot{x}\ddot{y}-\dot{y}\ddot{x})^2}$

$= |\dot{x}\ddot{y} - \dot{y}\ddot{x}|$.

Also, $|\vec{V}| = \sqrt{(\dot{x})^2 + (\dot{y})^2} \Rightarrow |\vec{v}|^3 = (\dot{x}^2 + \dot{y}^2)^{3/2}$.

There for $\kappa = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}}$.