Math 200 - Section B  Homework 2  KEY

1. $$\lim_{x,y\to 0,0} \cos\left(\frac{x^3-y^3}{x^2+y^2}\right) = \lim_{r\to 0} \cos\left(\frac{(r\cos\theta)^3-(r\sin\theta)^3}{(r\cos\theta)^2+(r\sin\theta)^2}\right)$$

$$= \lim_{r\to 0} \cos\left(\frac{r^{\cancel{3}}(\cos^3\theta-\sin^3\theta)}{\cancel{r^2}\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}}\right)$$

$$= \lim_{r\to 0} \cos\left(r(\cos^3\theta-\sin^3\theta)\right)$$

$$= \cos(0) = \boxed{1}$$

2. $$\frac{\partial^2}{\partial x^2}\left(2z^3-3(x^2+y^2)z\right) = \frac{\partial}{\partial x}(-6zx) = -6z$$

$$\frac{\partial^2}{\partial y^2}\left(2z^3-3(x^2+y^2)z\right) = \frac{\partial}{\partial y}(-6zy) = -6z$$

$$\frac{\partial^2}{\partial z^2}\left(2z^3-3(x^2+y^2)z\right) = \frac{\partial}{\partial z}\left(6z^2-3(x^2+y^2)\right) = 12z$$

So $\frac{\partial^2}{\partial x^2}f + \frac{\partial^2}{\partial y^2}f + \frac{\partial^2}{\partial z^2}f = -6z - 6z + 12z = 0$ ✓

3. $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$

$= (y)\cdot(-v^2 u^{-2}) + (x)\cdot(1) + (\frac{1}{z})\cdot(-\sin(u))$

$= -(u+v)(v^2/u^2) + (v^2/u) - \tan(u)$

at $(u,v) = (-1,2)$ this is $-(1)(4/1) + (4/-1) - \tan(-1) = -8 - \tan(-1)$

$\approx -6.4426$ *

$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$

$= (y)\cdot(2v/u) + (x)\cdot(1) + (\frac{1}{z})\cdot(0)$

$= (u+v)(2v/u) + (v^2/u)$

at $(u,v) = (-1,2)$ this is $(1)(4/-1) + (4/-1) = -8$

* $-7.9825$ if degrees used instead of radians

4. $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x - 2y, -2x + 6y)$

At point $P = (2,1)$, $\nabla f = (4-2, -4+6) = (2,2)$

This is the direction with the greatest rate of change. The rate of change in this direction is

$$|\nabla f| = |(2,2)| = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2} = 2.8\ldots < 3.$$

Since the direction with the greatest rate of change at point P has a rate of change less than 3, there can be no direction $\vec{u}$ in which the rate of change in f is equal to 3.

5. A unit vector in the direction of $\vec{v}$ (and $\nabla f$) is $\vec{u} = \frac{v}{|\vec{v}|} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}k$

Since $|\nabla f| = 2\sqrt{3}$, $\nabla f = 2\sqrt{3} \cdot \vec{u} = \boxed{2\hat{i} + 2\hat{j} - 2\hat{k}}$

6. $D_i f = f_x$
$D_j f = f_y$
$D_k f = f_z$

7. $\frac{\partial f}{\partial x} = 3x^2 + 3y$ zero when $y = -x^2$

$\frac{\partial f}{\partial y} = 3x + 3y^2$ zero when $x = -y^2$

- Critical points at : $(0,0)$ and $(-1,-1)$

$f_{xx} = 6x$

$f_{yy} = 6y$

$f_{xy} = 3$

At $(-1,-1)$ :

$$f_{xx} f_{yy} - f_{xy}^2 = (-6)(-6) - (3)^2 = 36 - 9 = 25 > 0$$

and

$$f_{xx} = -6 < 0$$

so f has a local maximum at $(-1,-1)$

The value at this point is $f(-1,-1) = 1$

At $(0,0)$:

$$f_{xx} f_{yy} - f_{xy}^2 = (0)(0) - (3)^2 = -9 < 0$$

so f has a saddle point at $(0,0)$

The value at this point is $f(0,0) = 0$.