QUIZ 3

(Math 200-Section B)

1. Find the limits: (a) $\lim_{(x,y)\to(1,0)} \frac{x \sin y}{x^2+1}$ (b) $\lim_{(x,y)\to(2,-4)} \frac{y+4}{x^2y - xy + 4x^2 - 4x}$ (4 pts)

Find the limits:

(a) $$\lim_{(x,y)\to(1,0)} \frac{x\sin(y)}{x^2+1} = \frac{1\sin(0)}{1^2+1} = \frac{1\cdot 0}{1+1} = \frac{0}{1} = \boxed{0}$$

(b) $$\lim_{(x,y)\to(2,-4)} \frac{y+4}{x^2y - xy + 4x^2 - 4x}$$

Now, $x^2y - xy + 4x^2 - 4x$
$= x(xy - y + 4x - 4)$
$= x(y(x-1) + 4(x-1))$
$= x\cdot(x-1)\cdot(y+4)$.

Plugging in, we get

$$= \lim_{(x,y)\to(2,-4)} \frac{y+4}{x(x-1)(y+4)}$$

$$= \lim_{(x,y)\to(2,-4)} \frac{1}{x(x-1)}$$

$$= \frac{1}{2(2-1)}$$

$$= \boxed{\frac{1}{2}}$$

2. Check whether the function $f(x,y) = \frac{1}{x^2 - y}$ is continuous at (-1,1). Can you describe all points of discontinuity of f? (4 pts)

$f(x,y) = \frac{1}{x^2-y}$. At $(-1,1)$, this is $\frac{1}{(-1)^2-1} = \frac{1}{0}$, undefined.

Therefore f is not continuous at $(-1,1)$. It is undefined, and therefore discontinuous wherever the denominator is 0, i.e., whenever $x^2 - y = 0$, ie everywhere on the parabola $y = x^2$. These are the only points of discontinuity.

3. By considering different paths of approach, show that the function $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ has no limit as $(x,y)\to(0,0)$. (4 pts)

Along the curve $y=mx$, $x\neq 0$, the function f has a constant value of

$$f(x,y)=f(x,mx)=\frac{x^2-(mx)^2}{x^2+mx^2}=\frac{x^2(1-m^2)}{x^2(1+m^2)}=\frac{1-m^2}{1+m^2}$$

Therefore $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x,y)=\lim_{x\to 0}\left(f(x,mx)\right)=\frac{1-m^2}{1+m^2}$

This limit varies with the path of approach* $\therefore$ the limit does not exist.

* for example if you approach (0,0) along the line $y=x$ (i.e. $m=1$) the limit is 0, but if you approach along $y=0$ (ie $m=0$), the limit is 1.

4. Find the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ of the function $f(x,y)=e^{2y}\cos(x-y)$. (4 pts)

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(e^{2y}\cos(x-y)\right)=\boxed{e^{2y}(-\sin(x-y))}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(e^{2y}\cos(x-y)\right)=(2e^{2y})(\cos(x-y))+(e^{2y})(-\sin(x-y)\cdot -1)$$
$$=\boxed{e^{2y}\left(2\cos(x-y)+\sin(x-y)\right)}$$

5. Find the value of $\dfrac{\partial z}{\partial x}$ at (1,1,1) if the equation $xy+z^3x-2yz=0$ defines z as a function of the two independent variables x and y, and the partial derivatives exist. (4 pts)

chain rule: $\frac{d}{dx}f(g(x))=f'(g(x))g'(x)$ i.e., $f'(g)\cdot\frac{dg}{dx}$ (for #5)

$xy+z^3x-2yz=0$

then $\frac{\partial}{\partial x}(xy+z^3x-2yz)=\frac{\partial}{\partial x}(0)$

$\Rightarrow y\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial x}(z^3x)-2y\frac{\partial}{\partial x}(z)=0$

$\Rightarrow y\cdot 1+\left(\frac{\partial}{\partial x}(z^3)\cdot x+z^3\cdot 1\right)-2y\cdot\frac{\partial z}{\partial x}=0$

(since y indep. of x, treat it as const. in the partial deriv. $\frac{\partial}{\partial x}$)

$\Rightarrow \left(3(z)^2\frac{dz}{dx}\right)\cdot x+z^3-2y\frac{\partial z}{\partial x}=-y$

$\Rightarrow 3xz^2\frac{\partial z}{\partial x}-2y\frac{\partial z}{\partial x}=-y-z^3$

$\Rightarrow (3xz^2-2y)\frac{\partial z}{\partial x}=-y-z^3$

$\Rightarrow \dfrac{\partial z}{\partial x}=\dfrac{-y-z^3}{3xz^2-2y}$ At (1,1,1), this is $\dfrac{-1-1}{3-2}=\boxed{-2}$