

Homework 3 - Key

MAT 200

1. Find the radius and interval of convergence for the power series:

a.

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)2^{n+1}}}{\frac{1}{n2^n}} = \lim_{n \rightarrow \infty} \frac{n2^n}{(n+1)2^{n+1}} = \lim_{n \rightarrow \infty} \frac{n}{(n+1)2} = \lim_{n \rightarrow \infty} \frac{n}{2n+2} = \frac{1}{2}$$

$$\text{So, } R = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

$$\text{For } x = 2: \sum_{n=1}^{\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n} \rightarrow \text{diverges}$$

$$\text{For } x = -2: \sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \rightarrow \text{converges}$$

Hence, $-2 \leq x < 2$.

b.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{n^n} = \lim_{n \rightarrow \infty} \frac{(n+1)^n}{n^n} (n+1) = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^n (n+1) = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n (n+1) \\ &= \lim_{n \rightarrow \infty} e(n+1) \\ &= \infty \end{aligned}$$

$$\text{So, } R = \frac{1}{\infty} = 0.$$

$$|x| < 0.$$

$$\text{For } x = 0: \sum_{n=1}^{\infty} n^n \cdot 0 = \sum_{n=1}^{\infty} 0 \rightarrow \text{converges}$$

Hence, $x = \{0\}$, i.e. the series converges only at $x = 0$.

$$\text{c. } \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \lim_{n \rightarrow \infty} \frac{n!3^{n+1}}{(n+1)!3^n} = \lim_{n \rightarrow \infty} \frac{3}{n+1} = 0$$

$$\text{So, } R = \frac{1}{0} = \infty.$$

$$|x| < \infty \Rightarrow -\infty < x < \infty$$

Hence, the series converges everywhere.

$$2. \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, -1 < x \leq 1.$$

For $x = 1$:

$$\begin{aligned} \ln(1+1) &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \\ &= 1 - .50 + 0.33 - 0.25 \\ &\approx 0.58 \end{aligned}$$

3. Term by term differentiation and integration:

$$\begin{aligned} \text{a. } \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ \frac{d}{dx} \sin x &= \frac{d}{dx} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right] = 1 - \frac{3x^2}{3!} - \frac{5x^4}{5!} + \frac{7x^6}{7!} - \dots \\ &= 1 - \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots \end{aligned}$$

$$\text{So, } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

For $x = \pi/6$:

$$\begin{aligned} \cos \frac{\pi}{6} &= 1 - \frac{\pi^2}{6 \cdot 2!} + \frac{\pi^4}{6^4 \cdot 4!} - \frac{\pi^6}{6^6 \cdot 6!} + \dots \\ &= 1 - \frac{\pi^2}{6^2 \cdot 2!} + \frac{\pi^4}{6^4 \cdot 4!} - \frac{\pi^6}{6^6 \cdot 6!} + \dots \\ \text{i.e., } \frac{\sqrt{3}}{2} &= 1 - \frac{\pi^2}{72} + \frac{\pi^4}{31104} - \frac{\pi^6}{33592320} + \dots \end{aligned}$$

$$\text{b. } \sum_{n=1}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \dots \text{ geometric series}$$

$a = 1$ and $r = -x^2$. Since $-1 < x < 1 \Rightarrow -1 < -x^2 < 0 \Rightarrow -1 < r < 1$.

$$\text{So, } 1 - x^2 + x^4 - x^6 = \frac{1}{1 - (-x^2)} dx = \frac{1}{1 + x^2}.$$

$$\text{Hence, } \int [1 - x^2 + x^4 - x^6] dx = \int \frac{1}{1 + x^2} dx \Rightarrow x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} = \tan^{-1} x + c$$

For $x = 0$: $0 = 0 + c \Rightarrow c = 0$.

$$\text{So, } \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}, -1 \leq x \leq 1.$$

For $x = 1$:

$$\tan^{-1} 1 = x - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \Rightarrow \pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7}.$$

i.e. $\pi \approx 2.90$