HOMEWORK 1
(Math 258)

1. Construct the truth table for each of the following compound propositions and determine whether they are equivalent. Is any one of them a tautology? (4pts)

   (a) $(q \to \neg p) \leftrightarrow (p \leftrightarrow q)$

p	q	$(q \to \neg p)$	$(p \leftrightarrow q)$	$(q \to \neg p) \leftrightarrow (p \leftrightarrow q)$
T	T	F	T	F
T	F	T	F	F
F	T	T	F	F
F	F	T	T	T

This is not a tautology.

   (b) $(p \to q) \oplus (p \leftrightarrow \neg q)$

p	q	$(p \to q)$	$(p \leftrightarrow \neg q)$	$(p \to q) \oplus (p \leftrightarrow \neg q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	T	F	T

This is a tautology.

2. Let *P(x)* be the statement "x = x^2." If the universe of discourse consists of all real integers, what is the truth value of the following? Explain.

   (a) $P(0)$
   0 = 0
   Truth value is **true**.

   (b) $P(-1)$
   -1 = 1;
   Truth value is **false**.

   (c) $\exists x P(x)$
   Let x = 1.
   Then 1 = 1.
   Truth value is **true**.

   (d) $\forall x P(x)$
   Let x = 2.
   Then 2 = 4.
   Truth value is **false**.

3. Determine the truth value of the following statement, if the universe of discourse consists of all real numbers. Explain. (4 pts)

   (a) $\forall x \exists y (x^2 = y)$
   A real number times a real number is always a real number.
   Truth value is **true**.

   (b) $\forall x \exists y (x = y^2)$
   If $x < 0$, you get imaginary numbers for y.
   Truth value is **false**.

   (c) $\forall x \exists y (xy = 0)$
   Let $y = 0$, then xy always equals 0.
   Truth value is **true**.

   (d) $\exists x \exists y (x + y \neq y + x)$
   Real numbers have commutativity of addition.
   Truth value is **false**.

4. Prove or disprove that product of a non-zero rational number and an irrational number is irrational.
   (*Hint:* Use the fact that the product of two rational numbers is rational.) (2 pts)

   Let a be rational, i.e. $a = \frac{b}{c}$, where b, c are rational numbers and $c \neq 0$.

   Let d be irrational.

   Notice that since $a \neq 0$ and $c \neq 0$, then $b \neq 0$. Hence, $\frac{c}{b}$ is also rational.

   Assume that ad is rational.

   Then, $(ad)(\frac{c}{b})$ is rational, by the Hint.

   But the product gives: $(ad)(\frac{c}{b}) = (\frac{b}{c}d)(\frac{c}{b}) = d$, which is irrational.

   Hence, we have a contradiction. So ad is irrational.

5. Determine which of the following sets is the power set of a set. For the ones that are power sets, write down the set. (2 pts)

   (α) $\emptyset$
   Not a power set.

   (b) $\{\emptyset, \{a\}\}$
   This is a power set.
   The set is: $\{a\}$.

(c) $\{\varnothing, \{a\}, \{\varnothing, a\}\}$
Not a power set.

(d) $\{\varnothing, \{a\}, \{b\}, \{a, b\}\}$
This is a power set.
The set is: $\{a, b\}$.

6. The **symmetric difference** of two sets A and B, denoted by $A \oplus B$, is the set containing those elements that belong to either A or B, but not both. (6pts)

(a) Show that is $A \oplus B = A$, then $B = \varnothing$.

Suppose B contains some element b.
If b is in A, then b is excluded from $A \oplus B$, so $A \oplus B \neq A$.
If b isn't in A, then b is in $A \oplus B$, so $A \oplus B \neq A$.
So, B must not contain any elements.

(b) Prove or disprove: $A \oplus C = B \oplus \mathrm{C}$, then $A = B$.

We will prove $A = B$. Only need to show $A \subset B$. By symmetry, we also get $B \subset A$.

Assume $A \oplus C = B \oplus C$, and let x be in A. Then there are two cases:

i. x is in C. Then x isn't in $A \oplus C$. By the hypothesis, x isn't in $B \oplus C$ either. So if x isn't in B, then x is in $B \oplus C$. But by the hypothesis, this can't be true. Therefore, x is in B.

ii. x isn't in C. Then x is in $A \oplus C$. By the hypothesis, x is in $B \oplus C$ also. So if x isn't in B, then x isn't in $B \oplus C$. But by the hypothesis, this can't be true. Therefore, x is in B.

Either way, x is in B.