Key

Final Exam

(Math 200, Fall 06)

Solve the following problems. Show all your work in the space under each problem.

1. Use partial fractions to find the sum of the series $\sum_{k=3}^{\infty} \frac{1}{k^2-k}$. (10 pts)

$a_k = \frac{1}{k^2-k} = \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$

$S_n = a_3 + a_4 + a_5 + \dots + a_n = \frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} + \cancel{\dots} + \cancel{\frac{1}{n-1}} - \frac{1}{n}$

$= \frac{1}{2} - \frac{1}{n}$

$\sum_{k=3}^{\infty} \frac{1}{k^2-k} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{2} - \frac{1}{n}\right] = \frac{1}{2}$ //

2. True or False: If $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Is the converse true? If your answer is "No" provide a counterexample. . (10 pts)

True

No, eg $\sum_{k=0}^{\infty} \frac{1}{k}$. Then, $\lim_{k\to\infty} \frac{1}{k} = 0$, but $\sum_{k=0}^{\infty} \frac{1}{k}$ diverges //

3. True or False: If $\sum_{k=0}^{\infty} |a_k|$ converges, then $\sum_{k=0}^{\infty} a_k$ converges. Is the converse true? If your answer is "No" provide a counterexample. (10 pts)

True

No, eg $\sum_{k=0} (-1)^k \frac{1}{k+1}$. Then, $\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1}$ converges, but $\sum_{k=0}^{\infty} \left|(-1)^k \frac{1}{k+1}\right| = \sum_{k=0}^{\infty} \frac{1}{k+1}$ diverges //

4. Show that the power series $\sum_{k=1}^{\infty} \frac{1}{k^2} x^k$ has interval of convergence [-1,1]. (10 pts)

$\frac{1}{R} = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{(k+1)^2} = \lim_{k\to\infty} \frac{k^2}{k^2+2k+1}$

$= 1$

ie, $R = 1$

So, $|x - 0| < R \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$

At $x = -1$: $\sum_{k=1}^{\infty} \frac{1}{k^2}(-1)^k$ converges (bec. it conv. absolutely)

At $x = 1$: $\sum_{k=1}^{\infty} \frac{1}{k^2}$ Converges (p-series with $p = 2 > 1$)

$\Rightarrow -1 \le x \le 1$ //

5. Find the Taylor series of $f(x) = e^{2x}$ at $x_0 = 1$. Make sure you include the n^{th} - term of the series. (10 pts)

$f(x) = e^{2x} \Rightarrow f(1) = e^2$

$f'(x) = 2e^{2x} \Rightarrow f'(1) = 2e^2$

$f''(x) = 2 \cdot 2 e^{2x} \Rightarrow f''(1) = 2 \cdot 2 e^2 = 2^2 e^2$

$f'''(x) = 2 \cdot 2 \cdot 2 e^{2x} \Rightarrow f'''(1) = 2 \cdot 2 \cdot 2 e^2 = 2^3 e^2$

$\vdots$

$f^{(n)}(x) = \underbrace{2 \cdot 2 \cdots 2}_{n} e^{2x} \Rightarrow f^{(n)}(1) = 2^n e^2$

$\rightarrow$ So, $f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \cdots + \frac{f^{(n)}}{n!}(x-1)^n + \cdots$

$= e^2 + \frac{2e^2}{1!}(x-1) + \frac{2^2 e^2}{2!}(x-1)^2 + \cdots + \frac{2^n e^2}{n!}(x-1)^n + \cdots$ //

6. Find the limits: (a) $\lim\limits_{(x,y)\to(\frac{\pi}{2},0)} \frac{\cos y - 2}{y - \sin x}$ (b) $\lim\limits_{(x,y)\to(2,1)} \frac{xy - 3y - 2x + 1}{x - 1}$ (20 pts)

(a) $\lim\limits_{(x,y)\to(\pi/2,0)} \frac{\cos y - 2}{y - \sin x} = \frac{\cos 0 - 2}{0 - \sin \pi/2} = \frac{1-2}{0-1} = \frac{-1}{-1} = 1$ //

(b) $\lim\limits_{(x,y)\to(2,1)} \frac{xy - 3y - 2x + 1}{x - 1} = \frac{2 \cdot 1 - 3 \cdot 1 - 2 \cdot 2 + 1}{2 - 1} = \frac{-4}{1} = -4$ //

7. By considering different paths of approach, show that the function $f(x,y) = \frac{x^2 + y}{y}$ has no limit as $(x,y) \to (0,0)$. (10 pts)

Consider the paths $y = kx^2$, $k \in \mathbb{R}$

Then, $\lim\limits_{\substack{(x,y)\to(0,0) \\ (\text{along } y = kx^2)}} \frac{x^2+y}{y} = \lim\limits_{(x,y)\to(0,0)} \frac{x^2 + kx^2}{kx^2} = \lim\limits_{(x,y)\to(0,0)} \frac{1+k}{k} = \frac{1+k}{k}$

Since the limit gives a different outcome for different k's this means that function has no limit as $(x,y) \to (0,0)$. //

8. Check whether the function $f(x,y) = \frac{1}{x^2 - y}$ is continuous at (1,-1). Can you describe all points of discontinuity of f? (10 pts)

$f(1,-1) = \frac{1}{1^2 - (-1)} = \frac{1}{2} = \lim\limits_{(x,y)\to(1,-1)} \frac{1}{x^2 - y}$; ie f is cont. at $(1,-1)$.

The points of discontinuity are all points that lie on the parabola $y = x^2$. //

9. Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the function $f(x,y) = e^{-x}\sin(x+y)$. (10 pts)

$\frac{\partial f}{\partial x} = e^{-x}\cos(x+y) - e^{-x}\sin(x+y)$

$\frac{\partial f}{\partial y} = e^{-x}\cos(x+y)$ //