Homework 1 - Key

(Math 200, A, B)

1. (a)

$$\int \sec^n x\, dx = \int \sec^{n-2}x \sec^2 x\, dx = \int \sec^{n-2}x \frac{1}{\cos^2 x} dx$$

$$= \int \sec^{n-2}x \frac{\sin^2 x + \cos^2 x}{\cos^2 x} dx = \int \sin x \sec^{n-2}x \frac{\sin x}{\cos^2 x} dx + \int \sec^{n-2}x\, dx$$

$$= \int \sin x \sec^{n-2}x \frac{1}{\cos x}\frac{\sin x}{\cos x} dx + \int \sec^{n-2}x\, dx$$

$$= \int \sin x \sec^{n-2}x \sec x \tan x\, dx + \int \sec^{n-2}x\, dx$$

$$\overset{\text{By Parts}}{=} \int \sin x\, d\left(\frac{\sec^{n-1}x}{n-1}\right) + \int \sec^{n-2}x\, dx$$

$$= \sin x \frac{\sec^{n-1}x}{n-1} - \int \frac{\sec^{n-1}x}{n-1} d(\sin x) + \int \sec^{n-2}x\, dx$$

$$= \frac{1}{n-1}\sin x \sec^{n-1}x - \frac{1}{n-1}\int \sec^{n-1}x \cos x + \int \sec^{n-2}x\, dx$$

$$= \frac{1}{n-1}\frac{\sin x}{\cos x}\sec^{n-2}x - \frac{1}{n-1}\int \sec^{n-2}x\, dx + \int \sec^{n-2}x\, dx$$

$$= \frac{1}{n-1}\tan x \sec^{n-2}x + \frac{n-2}{n-1}\int \sec^{n-2}x\, dx \quad //$$

(b)

$$\int \tan^n x\, dx = \int \tan^{n-2}x \tan^2 x\, dx = \int \tan^{n-2}x \frac{\sin^2 x}{\cos^2 x} dx$$

$$= \int \tan^{n-2}x \frac{1-\cos^2 x}{\cos^2 x} dx = \int \tan^{n-2}x \frac{1}{\cos^2 x} dx - \int \tan^{n-2}x\, dx$$

$$= \int \tan^{n-2}x \sec^2 x\, dx - \int \tan^{n-2}x\, dx$$

$$\underset{du = \sec^2 x\, dx}{\overset{u = \tan x}{=}} \int u^{n-2} du - \int \tan^{n-2}x\, dx$$

$$= \frac{1}{n-1} y^{n-1} - \int \tan^{n-2}x\, dx$$

$$= \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\, dx \quad //$$

2. (a)

(i) $$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \int_{-\pi}^{\pi} \frac{1}{2}\left[\sin(n-m)x + \sin(n+m)x\right] dx$$

$$= \begin{cases} n=m: & \int_{-\pi}^{\pi} \frac{1}{2}\left[0 + \sin(2n)x\right]dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(2n)x = \frac{1}{2}\left[-\cos(2n)x\right]_{-\pi}^{\pi} \\ & = \frac{1}{2}\left[-1+1\right] = 0 \\ n\neq m: & \int_{-\pi}^{\pi} \frac{1}{2}\left[\sin(n-m)x + \sin(n+m)x\right]dx \end{cases}$$

$$\parallel$$

$$\frac{1}{2}\left[-\frac{\cos(n-m)x}{n-m} - \frac{\cos(n+m)x}{n+m}\right]_{-\pi}^{\pi} = -\cancel{\frac{(-1)^{n-m}}{n-m}} - \cancel{\frac{(-1)^{n+m}}{n-m}} + \cancel{\frac{(-1)^{n-m}}{n-m}} + \cancel{\frac{(-1)^{n+m}}{n+m}}$$

$$= 0 \quad //$$

(ii) $$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos(n-m)x - \cos(n+m)x\right] dx$$

$$= \begin{cases} n=m: & \int_{-\pi}^{\pi} \frac{1}{2}\left[1 - \cos(n+m)x\right]dx = \left[\frac{1}{2}x\right]_{-\pi}^{\pi} - \left[\frac{\sin(n+m)x}{2(n+m)}\right]_{-\pi}^{\pi} = \pi - \underset{0}{\overset{\parallel}{0}} \\ n\neq m: & \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos(n-m)x - \cos(n+m)x\right]dx \end{cases}$$

$$\parallel$$

$$\frac{1}{2}\left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m}\right]_{-\pi}^{\pi} = 0 \quad //$$

(iii) $$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos(n-m)x + \cos(n+m)x\right] dx$$

$$= \begin{cases} n=m: & \int_{-\pi}^{\pi} \frac{1}{2}\left[1 + \cos(n+m)x\right]dx = \left[\frac{1}{2}x\right]_{-\pi}^{\pi} + \left[\frac{\sin(n+m)x}{n+m}\right]_{-\pi}^{\pi} = \underset{\pi}{\overset{\parallel}{\pi + 0}} \\ n\neq m: & \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos(n-m)x + \cos(n+m)x\right]dx \end{cases}$$

$$\parallel$$

$$\frac{1}{2}\left[\frac{\sin(n-m)x}{n-m} + \frac{\sin(n+m)x}{n+m}\right]_{-\pi}^{\pi} = 0 \quad //$$

(b) $$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\sum_{n=1}^{N} a_n \sin nx\right)\sin mx \, dx$$

$$= \sum_{n=1}^{N} \frac{a_n}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx \, dx$$

$$= \frac{a_n}{\pi}\cdot 0 + \frac{a_n}{\pi}\cdot 0 + \cdots + \underbrace{\frac{a_n}{\pi}\,\pi}_{n=m} + \frac{a_n}{\pi}0 + \cdots + \frac{a_n}{\pi}0$$

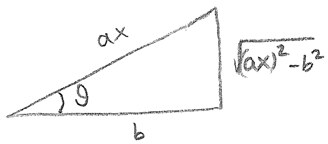
$$= a_n \quad //$$

3.

$$\int \frac{dx}{[(ax)^2-b^2]^{3/2}} \underset{a\,dx = b\sec\theta\tan\theta\,d\theta}{\overset{ax = b\sec\theta}{=}} \int \frac{\frac{b}{a}\sec\theta\tan\theta}{[b^2\sec^2\theta - b^2]^{3/2}}\,d\theta$$

$$= \int \frac{\frac{b}{a}\sec\theta\tan\theta}{b^3\tan^3\theta}\,d\theta = \frac{1}{ab^2}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

*



$$= \frac{1}{ab^2}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{ab^2}\int \csc\theta\cot\theta\,d\theta$$

$$= -\frac{1}{ab^2}\csc\theta$$

$$\overset{*}{=} -\frac{1}{ab^2}\frac{ax}{\sqrt{(ax)^2-b^2}} + C = -\frac{1}{b^2}\frac{x}{\sqrt{(ax)^2-b^2}} + C$$

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4. (a) $\tan(x/2) = t \Rightarrow \underbrace{\tan^2(x/2) + 1}_{\sec^2(x/2)} = t^2 + 1 \Rightarrow \sec^2(x/2) = t^2+1$

$\Rightarrow \cos(x/2) = \frac{1}{\sqrt{t^2+1}}$ //

Also, $\tan(x/2) = \frac{\sin(x/2)}{\cos(x/2)} \Rightarrow \sin(x/2) = \tan(x/2)\cos(x/2)$

$\Rightarrow \sin(x/2) = t\,\frac{1}{\sqrt{t^2+1}} \Rightarrow \sin(x/2) = \frac{t}{\sqrt{t^2+1}}$ //

(b) $\sin x = 2\sin(x/2)\cos(x/2) \overset{(a)}{=} 2\frac{t}{\sqrt{t^2+1}}\frac{1}{\sqrt{t^2+1}} \Rightarrow \sin x = \frac{2t}{t^2+1}$ //

$\cos x = \cos^2 x - \sin^2 x \overset{(a)}{=} \left(\frac{1}{\sqrt{t^2+1}}\right)^2 - \left(\frac{t}{\sqrt{t^2+1}}\right)^2 = \frac{1}{t^2+1} - \frac{t^2}{t^2+1}$

$= \frac{1-t^2}{t^2+1}$

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(c) Since $\tan(x/2) = t \Rightarrow \sec^2(x/2)\,\frac{1}{2}\,dx = 1\,dt$

$\Rightarrow dx = \frac{2\,dt}{\sec^2(x/2)} \Rightarrow dx = 2\cos^2(x/2)\,dt$

$\Rightarrow dx \overset{(a)}{=} \frac{2\,dt}{t^2+1}$

(d) $$\int \frac{dx}{3-5\sin x} \underset{(b),(c)}{\overset{t=\tan(x/2)}{=}} \int \frac{\frac{2\,dt}{t^2+1}}{3 - 5\cdot\frac{2t}{t^2+1}} = \int \frac{2\,dt}{3(1+t^2)-10t} = 2\int \frac{dt}{3t^2-10t+3}$$

$$= \frac{1}{4}\int \left[\frac{1}{t-3} - \frac{3}{3t-1}\right]dt = \frac{1}{4}\left[\ln|t-3| - \ln|3t-1|\right]$$

$$= \frac{1}{4}\ln\left|\frac{\tan(x/2)-3}{3\tan(x/2)-1}\right| + C$$

//

5. $$\int_0^\infty \frac{dx}{\sqrt{x}(1+x)} = \int_0^1 \frac{dx}{\sqrt{x}(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}(1+x)}$$

$$= \lim_{t\to 0^+}\int_t^1 \frac{dx}{\sqrt{x}(1+x)} + \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}(1+x)}$$

Now, $$\int \frac{dx}{\sqrt{x}(1+x)} \underset{dx = 2u\,du}{\overset{u=\sqrt{x}}{=}} 2\int \frac{du}{1+u^2} = 2\tan^{-1}u = 2\tan^{-1}\sqrt{x} + C$$

So, $$\lim_{t\to 0^+}\left[2\tan^{-1}\sqrt{x}\right]_t^1 + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{x}\right]_1^t$$

$$\parallel$$

$$\lim_{t\to 0^+}\left[2(\pi/4) - 2\tan^{-1}\sqrt{t}\right] + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{t} - 2(\pi/4)\right]$$

$$\parallel$$

$$\pi/2 - 0 + 2(\pi/2) - \pi/2$$

$$\parallel$$

$$\pi$$

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