HOMEWORK 1

(Math 200 A, B)

1. Establish the following reduction formulas: (20 pts)

(a) $\int \sec^n x\,dx = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2} x\,dx$

(b) $\int \tan^n x\,dx = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

(*Hint*: Write $\sec^n x$ as $\sec^{n-2} x \sec^2 x$ and make use of the identity $\sin^2 x + \cos^2 x = 1$. Same for $\tan^n x$).

2. (a) Given that n and m are positive integers, show the following: (20 pts)

(i) $\int_{-\pi}^{\pi} \sin nx \, \cos mx\,dx = 0$

(ii) $\int_{-\pi}^{\pi} \sin nx \, \sin mx\,dx = \begin{cases} 0 \,, & if \; n \neq m \\ \pi \,, & if \; n = m \end{cases}$

(iii) $\int_{-\pi}^{\pi} \cos nx \, \cos mx\,dx = \begin{cases} 0 \,, & if \; n \neq m \\ \pi \,, & if \; n = m \end{cases}$

(b) The (*finite*) **Fourier Series** of $f(x)$ is given by the sum:

$$f(x) = \sum_{k=1}^{N} a_k \sin kx = a_1 \sin x + a_2 \sin 2x + ... + a_N \sin Nx$$

Show that the *n*th coefficient a_n is given by the formula:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx$$

(*Hint*: Make use of the *product* identities for part (a)).

3. Evaluate the integral: $\int \frac{dx}{[(ax)^2 - b^2]^{3/2}}$ (10 pts)

4. The German mathematician Karl Weierstrass (1815-1897) noticed that the substitution $t = \tan(x/2)$ will convert any rational function of $\sin x$ and $\cos x$ into an ordinary rational function of t. (40 pts)

(a) If $t = \tan(x/2)$, $-\pi < x < \pi$, show that :

$$\sin(x/2) = \frac{t}{\sqrt{1+t^2}} \quad \text{and} \quad \cos(x/2) = \frac{1}{\sqrt{1+t^2}}$$

(b) Show that:

$$\sin x = \frac{2t}{1+t^2} \quad \text{and} \quad \cos x = \frac{1-t^2}{1+t^2}$$

(c) Show that: $dx = \frac{2}{1+t^2} dt$

(d) Use the above substitution to evaluate the integral: $\int \frac{dx}{3 - 5\sin x}$

5. Evaluate the improper integral: $\int_0^{\infty} \frac{1}{\sqrt{x}(1+x)} dx$ (10 pts)

(*Note*: The above integral is of both types, type 1 and type 2).