HOMEWORK 2

(Math 200 A, B)

1. What is the value of c if $\sum_{n=2}^{\infty}(1+c)^{-n} = 2$? (10 pts)

2. The **Fibonacci Sequence** is defined by the following equations: (30 pts)

$$f_1 = 1\,,\ f_2 = 1\,,\ f_n = f_{n-1} + f_{n-2}\ \ ,\ \text{where}\ \ n \geq 3\ .$$

Show the following: (a) $\dfrac{1}{f_{n-1} f_{n+1}} = \dfrac{1}{f_{n-1} f_n} - \dfrac{1}{f_n f_{n+1}}$

(b) $\sum_{n=2}^{\infty} \dfrac{1}{f_{n-1} f_{n+1}} = 1$

(c) $\sum_{n=2}^{\infty} \dfrac{f_n}{f_{n-1} f_{n+1}} = 2$

3. The power series for $\ln(1+x)$ is given by: (10 pts)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad ,\ -1 < x \leq 1.$$

Use this to approximate $\ln 2$ in two decimal places.

(*Note*: Just use the first four terms of the power series)

4. **Term-by-term differentiation and integration:** (20 pts)

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ **,** $-R < x < R$**, then:**

(i) $f(x)$ **is differentiable and** $f'(x) = \sum_{n=0}^{\infty} \dfrac{d}{dx}(a_n x^n)$ **,** $-R < x < R$

(ii) $f(x)$ **is integrable and** $\int f(x)dx = \sum_{n=0}^{\infty} a_n \dfrac{x^{n+1}}{n+1}$ **+** $\boldsymbol{C}$ **,** $-R < x < R$**.**

(a) Given that the power series of $\sin x$ is $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$ and using the information above, find the power series for $\cos x$ (write down only the first four terms). Use that to show that $\frac{\sqrt{3}}{2} = 1 - \frac{\pi^2}{72} + \frac{\pi^4}{31104} - \frac{\pi^6}{33592320} \cdots$.

(b) Using the geometric series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$, $-1 < x < 1$ and the information above, show that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, $-1 \le x \le 1$. Use that to approximate π in two decimal places.

(*Note*: Just use the first four terms of the power series)

5. The **Bessel Function** (of order 0) is a function expressed as a power series, named after the German mathematician and astronomer F. Bessel (1784-1846). This function came out of Bessel's solution to Kepler's equation describing planetary motion. It is defined as follows:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

(a) Find the domain of the function $J_0(x)$.
(*Hint*: Find the interval of convergence and explain that this is the same as the domain)

(b) Notice that $J_0(x) = \lim_{n \to \infty} S_n(x)$, where $S_n(x)$ are the partial-sum polynomials. Write down the general form of $S_n(x)$ (as a series), and use that to write down the first four polynomials $S_0(x)$, $S_1(x)$, $S_2(x)$ and $S_3(x)$ that approximate $J_0(x)$.

(c) Evaluate the integral $\int_0^1 J_0(x)\, dx$ (correct to 3 decimal places).
(*Hint*: Make use only of the first 3 terms of the power series)