HW2 - Key (Solutions were copied from the Textbook) "Calculus", J. Stewart

1

[illegible] $\sum_{n=2}^{\infty}(1+c)^{-n}$ is a geometric series with $a = (1+c)^{-2}$ and $r = (1+c)^{-1}$, so the series converges when $|(1+c)^{-1}| < 1 \iff |1+c| > 1 \iff 1+c>1$ or $1+c<-1 \iff c>0$ or $c<-2$. We calculate the sum of the series and set it equal to 2: $\dfrac{(1+c)^{-2}}{1-(1+c)^{-1}} = 2 \iff \left(\dfrac{1}{1+c}\right)^2 = 2 - 2\left(\dfrac{1}{1+c}\right) \iff$

$1 = 2(1+c)^2 - 2(1+c) \iff 2c^2 + 2c - 1 = 0 \iff c = \frac{-2\pm\sqrt{12}}{4} = \frac{\pm\sqrt{3}-1}{2}$. However, the negative root is inadmissible because $-2 < \frac{-\sqrt{3}-1}{2} < 0$. So $c = \frac{\sqrt{3}-1}{2}$.

2

64. (a) $$\text{RHS} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}} = \frac{f_n f_{n+1} - f_n f_{n-1}}{f_n^2 f_{n-1} f_{n+1}} = \frac{f_{n+1} - f_{n-1}}{f_n f_{n-1} f_{n+1}} = \frac{(f_{n-1}+f_n) - f_{n-1}}{f_n f_{n-1} f_{n+1}}$$
$$= \frac{1}{f_{n-1}f_{n+1}} = \text{LHS}$$

(b) $$\sum_{n=2}^{\infty} \frac{1}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}}\right) \quad \text{[from part (a)]}$$
$$= \lim_{n\to\infty}\left[\left(\frac{1}{f_1 f_2} - \frac{1}{f_2 f_3}\right) + \left(\frac{1}{f_2 f_3} - \frac{1}{f_3 f_4}\right) + \left(\frac{1}{f_3 f_4} - \frac{1}{f_4 f_5}\right) + \cdots + \left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}}\right)\right]$$
$$= \lim_{n\to\infty}\left(\frac{1}{f_1 f_2} - \frac{1}{f_n f_{n+1}}\right) = \frac{1}{f_1 f_2} - 0 = \frac{1}{1\cdot 1} = 1 \text{ because } f_n \to \infty \text{ as } n \to \infty.$$

(c) $$\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty}\left(\frac{f_n}{f_{n-1}f_n} - \frac{f_n}{f_n f_{n+1}}\right) \quad \text{(as above)}$$
$$= \sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right)$$
$$= \lim_{n\to\infty}\left[\left(\frac{1}{f_1} - \frac{1}{f_3}\right) + \left(\frac{1}{f_2} - \frac{1}{f_4}\right) + \left(\frac{1}{f_3} - \frac{1}{f_5}\right) + \left(\frac{1}{f_4} - \frac{1}{f_6}\right) + \cdots + \left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right)\right]$$
$$= \lim_{n\to\infty}\left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{f_n} - \frac{1}{f_{n+1}}\right) = 1 + 1 - 0 - 0 = 2 \text{ because } f_n \to \infty \text{ as } n \to \infty.$$

3

3. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, -1 < x \leq 1.$

For $x = 1$:

$$\ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$
$$= 1 - .50 + 0.33 - 0.25$$
$$\approx 0.58$$

4

4. Term by term differentiation and integration:

a. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

$$\frac{d}{dx}\sin x = \frac{d}{dx}[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots] = 1 - \frac{3x^2}{3!} - \frac{5x^4}{5!} + \frac{7x^6}{7!} - \ldots$$

$$= 1 - \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots$$

So, $\cos x = 1 - \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots$

For $x = \pi/6$:

$$\cos\frac{\pi}{6} = 1 - \frac{\frac{\pi}{6}^2}{2!} + \frac{\frac{\pi}{6}^4}{4!} - \frac{\frac{\pi}{6}^6}{6!} + \ldots$$

$$= 1 - \frac{\pi^2}{6^2 \cdot 2!} + \frac{\pi^4}{6^4 \cdot 4!} - \frac{\pi^6}{6^6 \cdot 6!} + \ldots$$

i.e., $\frac{\sqrt{3}}{2} = 1 - \frac{\pi^2}{72} + \frac{\pi^4}{31104} - \frac{\pi^6}{33592320} + \ldots$

b. $\sum_{n=1}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \ldots$ geometric series

$a = 1$ and $r = -x^2$. Since $-1 < x < 1 \Rightarrow -1 < -x^2 < 0 \Rightarrow -1 < r < 1$.

So, $1 - x^2 + x^4 - x^6 = \frac{1}{1-(-x^2)}dx = \frac{1}{1+x^2}$.

Hence, $\int[1 - x^2 + x^4 - x^6]dx = \int\frac{1}{1+x^2}dx \Rightarrow x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} = \tan^{-1}x + c$

.

For $x = 0$: $0 = 0 + c \Rightarrow c = 0$.

So, $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7}, -1 \le x \le 1$.

For $x = 1$:

$$\tan^{-1} 1 = x - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \Rightarrow \pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7}.$$

i.e. $\pi \approx 2.90$

FIGURE 2

5

a

EXAMPLE 3 Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

SOLUTION Let $a_n = (-1)^n x^{2n}/[2^{2n}(n!)^2]$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right|$$

$$= \frac{x^{2n+2}}{2^{2n+2}(n+1)^2(n!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}}$$

$$= \frac{x^2}{4(n+1)^2} \to 0 < 1 \qquad \text{for all } x$$

Thus, by the Ratio Test, the given series converges for all values of x. In other words, the domain of the Bessel function J_0 is $(-\infty, \infty) = \mathbb{R}$.

Recall that the sum of a series is equal to the limit of the sequence of partial sums. So when we define the Bessel function in Example 3 as the sum of a series we mean that, for every real number x,

$$J_0(x) = \lim_{n\to\infty} s_n(x) \qquad \text{where} \qquad s_n(x) = \sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{2^{2i}(i!)^2}$$

The first few partial sums are

$$s_0(x) = 1 \qquad s_1(x) = 1 - \frac{x^2}{4} \qquad s_2(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64}$$

$$s_3(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} \qquad s_4(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147{,}456}$$

(b) $$\int_0^1 J_0(x)\,dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right] dx = \int_0^1 \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots\right) dx$$

$$= \left[x - \frac{x^3}{3\cdot 4} + \frac{x^5}{5\cdot 64} - \frac{x^7}{7\cdot 2304} + \cdots\right]_0^1 = 1 - \frac{1}{12} + \frac{1}{320} - \frac{1}{16{,}128} + \cdots$$

Since $\frac{1}{16{,}128} \approx 0.000062$, it follows from The Alternating Series Estimation Theorem that, correct to three decimal places, $\int_0^1 J_0(x)\,dx \approx 1 - \frac{1}{12} + \frac{1}{320} \approx 0.920$.