KEY

TEST 2

(Math 258)

1. (i) Let $A = \{1, 2, 5, 7\}$ and $B = \{2, 3, 7, 9, 10\}$. Find: (40 pts)

(a) $A \cup B$

$\{1, 2, 3, 5, 7, 9, 10\}$

(b) $A \cap B$

$\{2, 7\}$

(c) $A - B$

$\{1, 5\}$

(d) $B - A$

$\{3, 9, 10\}$

(ii) Let $A = \{\varnothing\}$ and $B = \{\varnothing, \{\varnothing\}\}$. Find:

(a) $P(A)$

$\{\emptyset, \{\emptyset\}\}$

(b) $P(P(A))$

$\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

(c) $P(\varnothing)$

$\{\emptyset\}$

(d) $B \times B$

$\{(\emptyset, \emptyset), (\emptyset, \{\emptyset\}), (\{\emptyset\}, \emptyset), (\{\emptyset\}, \{\emptyset\})\}$

2. Determine which of the following sets is the power set of a set. For the ones that are power sets, write down the set. (10 pts)

(a) $\varnothing$

not a power set

(b) $\{\varnothing, \{a\}\}$

Yes, $A = \{a\}$

3. (a) Show that if $A \subseteq B$, then $A \cup B = B$. (10 pts)

Let $x \in A \cup B$

$\Rightarrow x \in A \vee x \in B$

$\overset{A \subseteq B}{\Rightarrow} x \in B \vee x \in B$, ie any arbitrary x in $A \cup B$ is in B, ie $A \cup B \subseteq B$.

(b) Find two sets A and B such that $A \in B$ and $A \subseteq B$.

Let

$A = \{\emptyset\}$

$B = \{\emptyset, \{\emptyset\}\}$

4. (a) Show that if $A \subseteq B$, then $P(A) \subseteq P(B)$. (10 pts)

Let $S \in P(A)$ $\Rightarrow S \subseteq A$ $\Rightarrow S \subseteq B$ $\Rightarrow S \in P(B)$ $\therefore P(A) \subseteq P(B)$

(b) Show that for any sets A and B, we have $A - B = A \cap \overline{B}$.

Let $x \in A - B$
$\Leftrightarrow x \in A \wedge x \notin B$
$\Leftrightarrow x \in A \wedge x \in \overline{B}$
$\Rightarrow x \in A \cap \overline{B}$ $\therefore A - B = A \cap \overline{B}$

5. Let A and B be subsets of a universal set U. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$. (10 pts)

(*Hint*: Establish the above using a Proof By Contradiction)

Let $x \in \overline{B}$. Assume $x \notin \overline{A}$ $\Rightarrow x \in A$ $\xRightarrow{A \subseteq B} x \in B$ $\Rightarrow x \notin \overline{B}$ $\downarrow$ $\rightarrow x \in \overline{A}$

Let $x \in A$. Assume $x \notin B$ $\Rightarrow x \in \overline{B}$ $\xRightarrow{\overline{B} \subseteq \overline{A}} x \in \overline{A}$ $\Rightarrow x \notin A$ $\downarrow$ $\rightarrow x \in B$

6. The **symmetric difference** of two sets A and B, denoted by $A \oplus B$, is the set containing those elements that belong to either A or B, but not in both. Answer the following: (10 pts)

(a) Show that $A \oplus U = \overline{A}$

Let $x \in A \oplus U$
$\Leftrightarrow x \notin (A \cap U) \wedge (x \in A \vee x \in U)$
$\Leftrightarrow x \notin A \wedge (x \in U)$
$\Leftrightarrow x \in \overline{A} \wedge x \in U$
$\Leftrightarrow x \in \overline{A}$ $\therefore A \oplus U = \overline{A}$

(b) If $A \oplus B = A$, what can you say about B? Explain.

B must be the $\emptyset$
Assume $B \neq \emptyset \Rightarrow \exists y \in B$
Case 1: if $y \in A$ $\Rightarrow y \notin A \oplus B$ $\Rightarrow y \notin A$ $\Rightarrow$ Contradiction
Case 2: if $y \notin A \Rightarrow y \in A \oplus B$ $\Rightarrow y \in A$ $\Rightarrow$ contradiction

7. The **successor** of a set A is the set $A \cup \{A\}$. Find the successors of the following sets: (10 pts)

(a) $\emptyset$ (b) $\{\emptyset\}$ (c) $\{\emptyset, \{\emptyset\}\}$

$\{\emptyset\}$ $\{\{\emptyset\}, \emptyset\}$ $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

How many elements does the successor of a set with n elements have?

$n+1$