TEST 3

(MATH 200 (A), Fall 06)

1. Use partial fractions to show that $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$. (10 pts)

Notice that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$.

So, $$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \lim_{n\to\infty} \left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + ... + \frac{1}{n} - \frac{1}{n+1} \right)$$
$$= \lim_{n\to\infty} \left(\frac{1}{1} - \frac{1}{n+1} \right)$$
$$= 1$$

2. For the geometric series $\sum_{n=0}^{\infty} e^{-3n}$, find the following: (15 pts)

(a) the first three terms (b) The ratio r (c) The sum of the series.

(a) $1, e^{-3}, e^{-6}$

(b) $r = e^{-3}$

(c) Sum $= \frac{a_1}{(1-r)} = \frac{1}{1-e^{-3}}$

3. Use the Integral Test to determine whether the series $\sum_{n=1}^{\infty} \frac{5n}{\frac{5}{2}n^2 - 1}$ converges or diverges. (10 pts)

Let $f(x) = \frac{5x}{\frac{5}{2}x^2 - 1}$, a positive, decreasing function of x for all $x \geq 1$.

$$\int_1^{\infty} f(x)dx = \lim_{b\to\infty} \int_1^{b} \frac{5x}{\frac{5}{2}x^2 - 1} dx$$

$$= \lim_{b\to\infty} \ln\left|\frac{5}{2}x^2 - 1\right|\Bigg|_1^b$$

$$= \lim_{b\to\infty}\left(\ln\left|\frac{5}{2}b^2 - 1\right| - \ln\left|\frac{5}{2} - 1\right|\right)$$

$$= \lim_{b\to\infty} \ln\left|\frac{5}{2}b^2 - 1\right| - \ln(3/2)$$

which diverges.

Since $a_n = \dfrac{5n}{\frac{5}{2}n^2 - 1} = f(n)$ for all $n \geq 1$, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \dfrac{5n}{\frac{5}{2}n^2 - 1}$ also diverges.

4. Use the Comparison Test to determine whether the series $\sum_{n=1}^{\infty} \dfrac{1+\sin n}{n^2}$ converges or diverges. (10 pts)

Since sine oscillates between –1 and 1, the numerator will oscillate between 0 and 2. We can see therefore see that the series has no negative terms, and that $\dfrac{1+\sin n}{n^2} \leq \dfrac{2}{n^2}$ for all n. Therefore, since the series $\sum_{n=1}^{\infty} \dfrac{2}{n^2}$ converges (it is a p-series), the series $\sum_{n=1}^{\infty} \dfrac{1+\sin n}{n^2}$ also converges.

5. Use the Ratio Test to determine whether the series $\sum_{n=1}^{\infty} \dfrac{e^n}{n!}$ converges or diverges. (15 pts)

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{e^{n+1}}{(n+1)!}}{\frac{e^n}{n!}} = \lim_{n\to\infty} \frac{e^1}{(n+1)} = 0 < 1, \text{ so the series converges.}$$

6. Use the n-th Root Test to determine whether the series $\sum_{n=1}^{\infty} \dfrac{n^n}{(2^n)^2}$ converges or diverges. (15 pts)

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty}\left(\frac{n^n}{(2^n)^2}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{n^n}{(2^2)^n}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{n}{4}\right) = \infty > 1$$

, i.e the series diverges.

7. Find the Taylor series of $f(x) = \frac{1}{x^2}$ at $x_0 = 1$. Make sure you include the n^{th} - term of the series. (15 pts)

f(1) = 1
f'(1) = -2x^{-3} = -2
f''(1) = 6x^{-4} = 6

$$\sum_{n=0}^{\infty} f(x) = 1 - 2(x - 1) + 3(x - 1)^2 - \ldots\ldots.$$

i.e. $\sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$

8. Find the *radius* of convergence around *x* = 1/2 and the interval of convergence for the power series: $\sum_{k=1}^{\infty} \frac{3^k}{k}(x - \frac{1}{2})^k$. (10 pts)

$$\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{3^{k+1}}{k+1}}{\frac{3^k}{k}} = \lim_{k\to\infty} \frac{3k}{k+1} = 3 \text{ , so } R = 1/\rho \Rightarrow R = 1/3 .$$

Hence, the interval of convergence is:

$$|x - 1/2| < 1/3 \Rightarrow -1/3 + 1/2 < x < 1/3 + 1/2$$
$$\Rightarrow 1/6 < x < 5/6$$

At the boundaries:

For *x* = 1/6 : $\sum_{k=1}^{\infty} \frac{3^k}{k}(-\frac{1}{3})^k = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$, which converges.

For *x* = 5/6 : $\sum_{k=1}^{\infty} \frac{3^k}{k}(\frac{1}{3})^k = \sum_{k=1}^{\infty} \frac{1}{k}$, which diverges.

So, the interval of convergence is: $1/6 \le x < 5/6$.