TEST 3

(MATH 200 (B), Fall 06)

1. Use partial fractions to show that $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}=1$. (10 pts)

Notice that $\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$.

So, $$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k(k+1)} &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=\lim_{n\to\infty}\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+....+\frac{1}{n}-\frac{1}{n+1}\right)\\ &=\lim_{n\to\infty}\left(\frac{1}{1}-\frac{1}{n+1}\right)\\ &=1\end{aligned}$$

2. For the geometric series $\sum_{n=0}^{\infty}e^{-3n}$, find the following: (15 pts)

(a) the first three terms (b) The ratio r (c) The sum of the series.

(a) $1, e^{-3}, e^{-6}$

(b) $r=e^{-3}$

(c) Sum $=\frac{a_1}{(1-r)}=\frac{1}{1-e^{-3}}$

3. Use the Integral Test to determine whether the series $\sum_{n=1}^{\infty}\frac{3n}{\frac{3}{2}n^2-1}$ converges or diverges. (10 pts)

Let $f(x)=\frac{3x}{\frac{3}{2}x^2-1}$, a positive, decreasing function of x for all $x\geq 1$.

$$\int_1^{\infty}f(x)dx=\lim_{b\to\infty}\int_1^{b}\frac{3x}{\frac{3}{2}x^2-1}dx$$

$$= \lim_{b\to\infty} \ln\left|\frac{3}{2}x^2 - 1\right|\Bigg|_1^b$$

$$= \lim_{b\to\infty}\left(\ln\left|\frac{3}{2}b^2 - 1\right| - \ln\left|\frac{3}{2} - 1\right|\right)$$

$$= \lim_{b\to\infty} \ln\left|\frac{3}{2}b^2 - 1\right| - \ln(1/2)$$

which diverges.

Since $a_n = \dfrac{3n}{\frac{3}{2}n^2 - 1} = f(n)$ for all $n \ge 1$, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \dfrac{3n}{\frac{3}{2}n^2 - 1}$ also diverges.

4. Use the Comparison Test to determine whether the series $\sum_{n=1}^{\infty} \dfrac{1+\cos n}{n^2}$ converges or diverges. (10 pts)

Since cosine oscillates between –1 and 1, the numerator will oscillate between 0 and 2. We can see therefore see that the series has no negative terms, and that $\dfrac{1+\cos n}{n^2} \le \dfrac{2}{n^2}$ for all *n*. Therefore, since the series $\sum_{n=1}^{\infty} \dfrac{2}{n^2}$ converges (it is a p-series), the series $\sum_{n=1}^{\infty} \dfrac{1+\cos n}{n^2}$ also converges.

5. Use the Ratio Test to determine whether the series $\sum_{n=1}^{\infty} \dfrac{(n+1)(n+2)}{n!}$ converges or diverges. (15 pts)

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+2)(n+3)}{(n+1)!}}{\dfrac{(n+1)(n+2)}{n!}} = \lim_{n\to\infty} \frac{n+3}{(n+1)^2} = 0 < 1$$, i.e. the series converges.

6. Use the *n*-th Root Test to determine whether the series $\sum_{n=1}^{\infty} \dfrac{(\ln n)^n}{n^n}$ converges or diverges. (15 pts)

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty}\left(\frac{(\ln n)^n}{n^n}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{\ln n}{n}\right).$$

Using L'Hôpital's Rule, we get:

$\rho = \lim_{n\to\infty}\left(\frac{\ln n}{n}\right) = \lim_{n\to\infty}\left(\frac{1/n}{1}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0 < 1$, so the series converges.

7. Find the Taylor series of $f(x) = \frac{1}{x^2}$ at $x_0 = 1$. Make sure you include the n^{th} - term of the series. (15 pts)

$f(1) = 1$
$f'(1) = -2x^{-3} = -2$
$f''(1) = 6x^{-4} = 6$

$$\sum_{n=0}^{\infty} f(x) = 1 - 2(x - 1) + 3(x - 1)^2 - \ldots\ldots.$$

i.e. $\sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$

8. Find the *radius* of convergence around $x = 1/2$ and the interval of convergence for the power series: $\sum_{k=1}^{\infty} \frac{3^k}{k}(x-\frac{1}{2})^k$. (10 pts)

$$\rho = \lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{\frac{3^{k+1}}{k+1}}{\frac{3^k}{k}} = \lim_{k\to\infty}\frac{3k}{k+1} = 3 \text{ , so } R = 1/\rho \Rightarrow R = 1/3 .$$

Hence, the interval of convergence is:

$$\begin{aligned} |x - 1/2| < 1/3 \ &\Rightarrow\ -1/3 + 1/2 < x < 1/3 + 1/2 \\ &\Rightarrow 1/6 < x < 5/6 \end{aligned}$$

At the boundaries:

For $x = 1/6$: $\sum_{k=1}^{\infty}\frac{3^k}{k}(-\frac{1}{3})^k = \sum_{k=1}^{\infty}(-1)^k\frac{1}{k}$, which converges.

For $x = 5/6$: $\sum_{k=1}^{\infty}\frac{3^k}{k}(\frac{1}{3})^k = \sum_{k=1}^{\infty}\frac{1}{k}$, which diverges.

So, the interval of convergence is: $1/6 \le x < 5/6$.