Midterm Exam

Math 258
(Fall 2006)

Solve the following problems. Show all your work in the space under each problem.

1. Determine the truth value of the following statements, if the universe of discourse consists of all integers. Explain. (20 pts)

(a) $\forall n(n^2 \geq n)$

T (equality is obtained when $n=0$ or $n=1$)

(b) $\exists n(n^2 = 2)$

F ($\nexists$ integer st $n^2=2$)

(c) $\exists n \forall m(n+m=m)$

T ($\exists\, n=0$ st $0+m=m$ $\forall m$)

(d) $\exists n \exists m[(n-m=3) \wedge (n+m=2)]$

F ($\nexists$ integer sol's to the system. Indeed, $\left.\begin{matrix} n-m=3 \\ n+m=2 \end{matrix}\right\} \overset{+}{\Rightarrow} 2n=5 \Downarrow n=5/2 \notin \mathbb{Z}$)

2. (a) Prove that if x is irrational, and $x \geq 0$, then $\sqrt{x}$ is irrational. (15 pts)

Assume $\sqrt{x}$ is not irrat. $\Rightarrow \sqrt{x} = \frac{a}{b}$, $a,b \in \mathbb{Z}$

$\overset{\wedge 2}{\Rightarrow} x = a^2/b^2 \in \mathbb{Z}$ $\lightning$ (contradiction), ie $\sqrt{x}$ is irrat.

(b) Prove or disprove that the product of two irrational numbers is irrational.
(*Note*: If you are disproving, a counterexample should be sufficient)

Take $x=\sqrt{2}$ and $y=\sqrt{2}$ irrat's.
Then, $x \cdot y = \sqrt{2}\sqrt{2} = 2$ which is not irrat.

(c) Prove that there exists an integer m such that $m^2 > 10^{1000}$.
(*Hint*: Consider giving a constructive proof)

Take $m = 10^{500}+1$. Then, $m^2 = (10^{500}+1)^2 > (10^{500})^2 = 10^{1000}$.
ie $\exists\, m = 10^{500}+1$ with the desired property.

3. Show that if A and B are sets, then $A-(A-B) = A \cap B$. (15 pts)

Let $x \in A-(A-B) \Rightarrow x \in A \wedge x \notin A-B$
$\Rightarrow x \in A \wedge x \in B$. ($x \in B$, bec. if $x \notin B$ since $x \in A$ $\Rightarrow x \in A-B$ $\lightning$)
$\Rightarrow x \in A \cap B$.

Let $x \in A \cap B \Rightarrow x \in A \wedge x \in B$
$\Rightarrow x \notin A-B$ (bec. if $x \in A-B \Rightarrow x \in A \wedge x \notin B$ $\lightning$)
ie $x \in A$ and $x \notin A-B$
$\Rightarrow x \in A-(A-B)$.

4. Show that the following function is a bijection: $f: \mathbf{R} \to \mathbf{R}$ (10 pts)
$f(x) = 3x+5$

(i) 1-1 : Let $f(x_1) = f(x_2) \Rightarrow 3x_1 + 5 = 3x_2 + 5$
$\Rightarrow 3x_1 = 3x_2$
$\Rightarrow x_1 = x_2$, ie f is 1-1.

(ii) onto : $\forall y \ \exists x = \frac{y-5}{3}$ st $f(x) = 3\,\frac{y-5}{3} + 5 = y - 5 + 5 = y$, ie f is onto

Since f is 1-1 and onto, then f is a bijection.

5. (a) If $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ and $AB = 0$, show that $A = 0$. (20 pts)

Let $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then, $AB = 0 \Rightarrow \begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x & 2x-y \\ z & 2z-w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow x = 0,\ 2x - y = 0,\ z = 0,\ 2z - w = 0$

ie, $A = 0$ $\Rightarrow x = 0, y = 0, z = 0, w = 0.$

(b) Given that A and B are $n \times n$ matrices, show that $(A+B)(A-B) = A^2 - B^2$ if and only if $AB = BA$.

Let $AB = BA$. Then, $(A+B)(A-B) = A^2 - AB + BA - B^2 = A^2 - \cancel{BA} + \cancel{BA} - B^2$
$= A^2 - B^2$.

Let $(A+B)(A-B) = A^2 - B^2 \Rightarrow A^2 - AB + BA - B^2 = A^2 - B^2$
$\Rightarrow -AB + BA = 0$
$\Rightarrow AB = BA$.

6. (a) Show that if a, b and c are integers with $c \neq 0$, such that $ac|bc$, then $a|b$. (10 pts)

$ac|bc \Rightarrow bc = k(ac)$, some k.
$\Rightarrow b\cancel{c} = (ka)\cancel{c} \Rightarrow b = ka \Rightarrow a|b$.

(b) Prove or disprove that if $a|bc$, where a, b and c are positive integers, then $a|b$ or $a|c$.

Take $a = 4$, $b = 2$ and $c = 2$.
Then, $4|2\cdot2$, but $4\nmid 2$ and $4 \nmid 2$.

7. (a) Let $a = 3^7 5^3 7^3$ and $b = 2^{11} 3^5 5^9$. Find the **gcd**(a,b) and the **lcm**(a,b). (10 pts)

$a = 2^0 3^7 5^3 7^3$, $b = 2^{11} 3^5 5^9 7^0$

$\gcd(a,b) = 2^0 3^5 5^3 7^0 = 3^5 5^3$, $\operatorname{lcm}(a,b) = 2^{11} 3^7 5^9 7^3$.

(b) If the product of two integers is $2^7 3^8 5^2 7^{11}$ and their **gcd** is $2^3 3^4 5$, what is their **lcm** ?

We know that $ab = \gcd(a,b)\cdot \operatorname{lcm}(a,b)$

$\Rightarrow 2^7 3^8 5^2 7^{11} = 2^3 3^4 5 \cdot \operatorname{lcm}(a,b) \Rightarrow \operatorname{lcm}(a,b) = 2^4 3^4 5\, 7^{11}$.