Midterm Exam

Math 258
(Fall 2006)

Solve the following problems. Show all your work in the space under each problem.

1. Determine the truth value of the following statements, if the universe of discourse consists of all integers. Explain. (20 pts)

   (a) $\forall n(n^2 \geq n)$ (b) $\exists n(n^2 = 2)$

   (c) $\exists n \forall m(n + m = m)$ (d) $\exists n \exists m[(n - m = 3) \wedge (n + m = 2)]$

2. (a) Prove that if x is irrational, and $x \geq 0$, then $\sqrt{x}$ is irrational. (15 pts)

   (b) Prove or disprove that the product of two irrational numbers is irrational.
   (*Note*: If you are disproving, a counterexample should be sufficient)

   (c) Prove that there exists an integer m such that $m^2 > 10^{1000}$.
   (*Hint*: Consider giving a constructive proof)

3. Show that if A and B are sets, then $A - (A - B) = A \cap B$. (15 pts)

4. Show that the following function is a bijection: $f : \mathbf{R} \to \mathbf{R}$, $f(x) = 3x + 5$ (10 pts)

5. (a) If $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ and $AB = 0$, show that $A = 0$. (20 pts)

(b) Given that A and B are $n \times n$ matrices, show that $(A+B)(A-B) = A^2 - B^2$ if and only if $AB = BA$.

6. (a) Show that if a, b and c are integers with $c \neq 0$, such that $ac|bc$, then $a|b$. (10 pts)

(b) Prove or disprove that if $a|bc$, where a, b and c are positive integers, then $a|b$ or $a|c$.

7. (a) Let $a = 3^7 5^3 7^3$ and $b = 2^{11} 3^5 5^9$. Find the **gcd**(a,b) and the **lcm**(a,b). (10 pts)

(b) If the product of two integers is $2^7 3^8 5^2 7^{11}$ and their **gcd** is $2^3 3^4 5$, what is their **lcm** ?