TEST 2

(MATH 200 (A), Fall 06)

1. Evaluate the following integral: $\int 2\sin 6\theta \cos 2\theta \, d\theta$. (10 pts)

$$\int 2 \cdot \frac{1}{2}\left[\sin(6\theta - 2\theta) + \sin(6\theta + 2\theta)\right] d\theta = \int (\sin 4\theta + \sin 8\theta)\, d\theta$$
$$= -\frac{\cos 4\theta}{4} - \frac{\cos 8\theta}{8} + C$$

2. Show that: $\int_0^{\pi/4} \tan^4 x \sec^4 x \, dx = \frac{12}{35}$. (10 pts)

$$\int \tan^4 x \sec^4 x \, dx = \int \tan^4 \underset{\sim\sim}{\sec^2 x}\, \sec^2 x \, dx = \int \tan^4 x \,\underset{\sim\sim\sim}{(\tan^2 x + 1)} \sec^2 x \, dx$$
$$\underset{du = \sec^2 x\, dx}{\overset{u = \tan x}{=}} \int u^4 (u^2 + 1)\, du = \int (u^6 + u^4)\, du$$
$$= \frac{u^7}{7} + \frac{u^5}{5} = \frac{\tan^7\theta}{7} + \frac{\tan^5\theta}{5} + C$$

So, $\int_0^{\pi/4} \tan^4 x \sec^4 x \, dx = \left[\frac{\tan^7\theta}{7} + \frac{\tan^5\theta}{5}\right]_0^{\pi/4} = \left[\frac{1}{7} + \frac{1}{5}\right] - [0 + 0]$
$$= \frac{12}{35}$$

3. Using the identity $\cos^2 x + \sin^2 x = \cos 2x$, show that: $\int \frac{1 - \tan^2 x}{\sec^2 x} dx = \frac{1}{2}\sin 2x + C$.

(*Hint*: This should take you no more than 4 steps) (10 pts)

$$\int \frac{1 - \tan^2 x}{\sec^2 x} dx = \int \left(\frac{1}{\sec^2 x} - \frac{\tan^2 x}{\sec^2 x}\right) dx$$
$$= \int (\cos^2 x - \sin^2 x)\, dx$$
$$= \int \cos 2x \, dx$$
$$= \frac{\sin 2x}{2} + C$$

4. Evaluate the following integral: $\int \frac{dx}{\sqrt{x^2 - 2x + 5}}$. (10 pts)

$$\int \frac{dx}{\sqrt{x^2 - 2x + 5}} = \int \frac{dx}{\sqrt{(x-1)^2 + 4}} \underset{dx = 2\sec^2\theta}{\overset{x - 1 = 2\tan\theta}{=}} \int \frac{2\sec^2\theta}{\sqrt{4\tan^2\theta + 4}} d\theta$$
$$= \int \frac{\cancel{2}\sec^{\cancel{2}}\theta \, d\theta}{\cancel{2}\cancel{\sec\theta}}$$
$$= \int \sec\theta \, d\theta$$
$$= \ln|\sec\theta + \tan\theta|$$
$$\overset{*}{=} \ln\left|\sqrt{\left(\frac{x-1}{2}\right)^2 + 1} + \frac{x-1}{2}\right| + C$$

$*$ $x - 1 = 2\tan\theta$

$\tan\theta = \frac{x-1}{2}$

But $\tan^2\theta + 1 = \sec^2\theta$

$\Rightarrow \sec\theta = \sqrt{\left(\frac{x-1}{2}\right)^2 + 1}$

5. Evaluate the following integral: $\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$. (20 pts)

(*Hint*: Here you will need to use partial fractions)

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2+4}$$

$$\Rightarrow \quad 2x^2 - x + 4 = A(x^2+4) + (Bx+C)x$$

For $x = 0 \Rightarrow 4 = 4A \Rightarrow \boxed{A = 1}$

For $x = 1 \Rightarrow 5 = 5 + B + C \Rightarrow B = -C$

For $x = -1 \Rightarrow 7 = 5 + (-B + C)(-1) \Rightarrow B - C = 2$

$\Rightarrow \boxed{C = -1}$, $\boxed{B = 1}$

So,
$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \left(\frac{1}{x} + \frac{x - 1}{x^2+4}\right) dx$$
$$= \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$
$$= \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{\tan^{-1}(x/2)}{2} + C$$

6. Evaluate (if possible) the following improper integrals: (30 pts)

(a) $\int_4^{\infty} e^{-y/2}\,dy$

$$\int_4^{\infty} e^{-y/2}\,dy = \lim_{t\to\infty} \int_4^t e^{-y/2}\,dy$$
$$= \lim_{t\to\infty} \left[2e^{-y/2}\right]_4^t$$
$$= \lim_{t\to\infty} \left[2e^{-t/2} - 2e^{-2}\right]$$
$$= 0 - 2e^{-2}$$
$$= -2/e^2$$

(b) $\int_0^3 (x-2)^{-4/3}\,dx$

$$\int_0^3 (x-2)^{-4/3}\,dx = \int_0^2 (x-2)^{-4/3}dx + \int_2^3 (x-2)^{-4/3}dx$$
$$= \lim_{t\to 2} \int_0^t (x-2)^{-4/3}dx + \lim_{t\to 2} \int_t^3 (x-2)^{-4/3}dx$$
$$= \lim_{t\to 2}\left[-3(x-2)^{-1/3}\right]_0^t + \lim_{t\to 2}\left[-3(x-2)^{-1/3}\right]_t^3$$
$$= \lim_{t\to 2}\left[\frac{-3}{\sqrt[3]{t-2}} + \frac{3}{\sqrt[3]{-2}}\right] + \lim_{t\to 2}\left[-3 + \frac{3}{\sqrt[3]{t-2}}\right]$$
$$= \infty + \infty$$
$$= \infty$$

7. Use the Comparison Test to determine whether the integral $\int_1^{\infty} \frac{2 + e^{-x}}{x}\,dx$ is convergent or divergent. (10 pts)

$\frac{2 + e^{-x}}{x} > \frac{2}{x}$, since $e^{-x} = \frac{1}{e^x} > 0$

But $\int_1^{\infty} \frac{2}{x}\,dx$ diverges, bec. $\int_1^{\infty} \frac{2}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{2}{x}\,dx = \lim_{t\to\infty}\left[2\ln|x|\right]_1^t$

$= \lim_{t\to\infty}\left[2\ln t - 0\right]$

$= \infty$

Hence, $\int_1^{\infty} \frac{2 + e^{-x}}{x}\,dx$ diverges too, by the Comp. Test.