TEST 2

(MATH 200 (B), Fall 06)

1. Evaluate the following integral: $\int 2\cos 7\theta \cos 3\theta \, d\theta$. (10 pts)

$$\int 2\cdot\frac{1}{2}\left[\cos(7\theta-2\theta)+\cos(7\theta+2\theta)\right]dx = \int(\cos 5\theta + \cos 9\theta)\,d\theta$$

$$= \frac{\sin 5\theta}{5} + \frac{\sin 9\theta}{9} + C$$

2. Show that: $\int_0^{\pi/4} \tan^4 x \sec^4 x \, dx = \frac{12}{35}$. (10 pts)

$$\int \tan^4 x \sec^4 x\, dx = \int \tan^4 x \underbrace{\sec^2 x}\sec^2 x\, dx = \int \tan^4 x\underbrace{(\tan^2 x + 1)}\sec^2 x\, dx$$

$$\underset{du=\sec^2 x\,dx}{\overset{u=\tan x}{=}} \int u^4(u^2+1)\,du = \int (u^6+u^4)\,du$$

$$= \frac{u^7}{7} + \frac{u^5}{5} = \frac{\tan^7\theta}{7} + \frac{\tan^5\theta}{5} + C$$

So, $$\int_0^{\pi/4} \tan^4 x \sec^4 x\, dx = \left[\frac{\tan^7\theta}{7} + \frac{\tan^5\theta}{5}\right]_0^{\pi/4} = \left[\frac{1}{7}+\frac{1}{5}\right] - [0+0] = \frac{12}{35}$$

3. Using the identity $\cos^2 x - \sin^2 x = \cos 2x$, show that: $\int \frac{1-\tan^2 x}{\sec^2 x}dx = \frac{1}{2}\sin 2x + C$.

(*Hint*: This should take you no more than 4 steps) (10 pts)

$$\int \frac{1-\tan^2 x}{\sec^2 x}dx = \int\left(\frac{1}{\sec^2 x} - \frac{\tan^2 x}{\sec^2 x}\right)dx$$

$$= \int(\cos^2 x - \sin^2 x)\,dx$$

$$= \int \cos 2x\, dx$$

$$= \frac{\sin 2x}{2} + C$$

4. Evaluate the following integral: $\int \frac{3}{\sqrt{8-6x-9x^2}}dx$. (10 pts)

$$\int \frac{3}{\sqrt{8-6x-9x^2}}dx = \int \frac{3}{\sqrt{9-(1+3x)^2}}dx \underset{3\,dx = 3\cos\theta\, d\theta}{\overset{1+3x=3\sin\theta}{=}} \int \frac{3\cos\theta\, d\theta}{\sqrt{9-9\sin^2\theta}}$$

$$= \int \frac{\cancel{3\cos\theta}}{\cancel{3\cos\theta}}\,d\theta$$

$$= \int d\theta$$

$$= \theta$$

$$\overset{*}{=} \sin^{-1}\left(\frac{1+3x}{3}\right) + C$$

* $1+3x = 3\sin\theta$

$\sin\theta = \frac{1+3x}{3}$

$\theta = \sin^{-1}\left(\frac{1+3x}{3}\right)$

5. Evaluate the following integral: $\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$. (20 pts)

(*Hint*: Here you will need to use partial fractions)

$$\frac{2x^2 - x - 4}{x^3 + 4x} = \frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2+4}$$

$$\Rightarrow 2x^2 - x + 4 = A(x^2+4) + (Bx + C)x$$

For $x = 0 \Rightarrow 4 = 4A \Rightarrow \boxed{A = 1}$

For $x = 1 \Rightarrow 5 = 5 + B + C \Rightarrow B = -C$

For $x = -1 \Rightarrow 7 = 5 + (-B + C)(-1) \Rightarrow B - C = 2$

$\Rightarrow \boxed{C = -1}$, $\boxed{B = 1}$

So, $$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx = \int \left(\frac{1}{x} + \frac{x - 1}{x^2+4} \right) dx$$
$$= \int \frac{1}{x} dx + \int \frac{x}{x^2+4} dx - \int \frac{1}{x^2+4} dx$$
$$= \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{\tan^{-1}(x/2)}{2} + C$$

6. Evaluate (if possible) the following improper integrals: (30 pts)

(a) $\int_1^\infty \frac{\ln x}{x^2} dx$

$$\int_1^\infty \frac{\ln x}{x^2} dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2} dx$$
$$= \lim_{t\to\infty} \int_1^t \ln x \, d\left(-\frac{1}{x}\right)$$
$$= \lim_{t\to\infty} \left(\left[\ln x \left(-\frac{1}{x}\right)\right]_1^t + \int_1^t \frac{1}{x} d(\ln x) \right)$$
$$= \lim_{t\to\infty} \left(-\frac{\ln t}{t} - 0 + \int_1^t \frac{1}{x^2} dx \right)$$
$$= \lim_{t\to\infty} \left(-\frac{\ln t}{t} - \left[\frac{1}{x}\right]_1^t \right)$$
$$= \lim_{t\to\infty} \left(-\frac{\ln t}{t} - \frac{1}{t} + 1 \right) \overset{LH}{=} -0 - 0 + 1 = 1$$

(b) $\int_0^3 (x-2)^{-4/3} dx$

$$\int_0^3 (x-2)^{-4/3} dx = \int_0^2 (x-2)^{-4/3} dx + \int_2^3 (x-2)^{-4/3} dx$$
$$= \lim_{t\to 2} \int_0^t (x-2)^{-4/3} dx + \lim_{t\to 2} \int_t^3 (x-2)^{-4/3} dx$$
$$= \lim_{t\to 2} \left[-3(x-2)^{-1/3}\right]_0^t + \lim_{t\to 2} \left[-3(x-2)^{-1/3}\right]_t^3$$
$$= \lim_{t\to 2} \left[\frac{-3}{\sqrt[3]{t-2}} + \frac{3}{\sqrt[3]{-2}} \right] + \lim_{t\to 2} \left[-3 + \frac{3}{\sqrt[3]{t-2}} \right]$$
$$= \infty + \infty$$
$$= \infty$$

7. Use the Comparison Test to determine whether the integral $\int_1^\infty \frac{\cos^2 x}{1+x^2} dx$ is convergent or divergent. (10 pts)

$$\frac{\cos^2 x}{1 + x^2} \le \frac{1}{1+x^2} < \frac{1}{x^2}, \text{ ie } \frac{\cos^2 x}{1+x^2} < \frac{1}{x^2}$$

But $\int_1^\infty \frac{1}{x^2} dx$ converges, bec. $\int_1^\infty \frac{1}{x^2} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2} dx = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t$
$$= \lim_{t\to\infty} \left[-\frac{1}{t} + 1\right]$$
$$= 1$$

Hence, $\int_1^\infty \frac{\cos^2 x}{1+x^2}$ converges too, by the Comp. Test.