MAT 140 **Solutions Problems Week 4** **Fall 2007**

1. Given $A=\begin{pmatrix}3 & -1 & 0\\ 1 & 2 & -4\end{pmatrix}$ and $B=\begin{pmatrix}1 & -1 & 1\\ -2 & 0 & 6\end{pmatrix}$ find

(1) $-A=\boxed{\begin{pmatrix}-3 & 1 & 0\\ -1 & -2 & 4\end{pmatrix}}$ **(2)** $A-B=\boxed{\begin{pmatrix}2 & 0 & -1\\ 3 & 2 & -10\end{pmatrix}}$ **(3)** $A+2B=\boxed{\begin{pmatrix}5 & -3 & 2\\ -3 & 2 & 8\end{pmatrix}}$

2. Given $A=\begin{pmatrix}1 & 3 & 2\end{pmatrix}$ and $B=\begin{pmatrix}5\\ 6\\ 4\end{pmatrix}$ find

(1) $AB=\boxed{\begin{pmatrix}31\end{pmatrix}}$ **(2)** $BA=\boxed{\begin{pmatrix}5 & 15 & 10\\ 6 & 18 & 12\\ 4 & 12 & 8\end{pmatrix}}$ **(3)** Can you find A^2? **No**, it is not possible to multiply a 1×3 matrix with a 1×3 matrix [the 'inner' dimensions don't match]

3. Given $A=\begin{pmatrix}6 & 9\\ -4 & -6\end{pmatrix}$ and $B=\begin{pmatrix}1 & 2\\ -1 & 0\end{pmatrix}$ find

(1) $A^2=\boxed{\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}}$ **(2)** $AB=\boxed{\begin{pmatrix}-3 & 12\\ 2 & -8\end{pmatrix}}$ **(3)** $BA=\boxed{\begin{pmatrix}-2 & -3\\ -6 & -9\end{pmatrix}}$

(4) $B^2-A-2I_2=\boxed{\begin{pmatrix}-9 & -7\\ 3 & 2\end{pmatrix}}$

4. Show that $(A-B)(A+B)=A^2-B^2$ iff $AB=BA$

Proof: By the distributive properties of matrices [$C(A+B)=CA+CB$ and $(A-B)C=AC-BC$] we find that

$$\begin{aligned}(A-B)(A+B)&=(A-B)A+(A-B)B\\ &=AA-BA+AB-BB\\ &=A^2-B^2\ +\ (AB-BA)\end{aligned}$$

Hence $(A-B)(A+B)\ =\ A^2-B^2$ iff $AB-BA=0$ $\square$

5. Show that the matrix $A = \begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix}$ doesn't have an inverse.

Proof: If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ were an inverse of A then $\begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, but this cannot happen since

$$\begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathbf{0} & 0 \\ a+3c & b+3d \end{pmatrix} \neq \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 1 \end{pmatrix} \qquad \square$$

6. Find the inverse of the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

(a) $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \dfrac{1}{-1}\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}}$

or

(b) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} c & d \\ a+c & b+d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{cases} c = 1 \\ d = 0 \\ a+c = 0 \\ b+d = 1 \end{cases}$

Hence $a = -1$, $b = 1$, $c = 1$ and $d = 0$ $\therefore$ $A^{-1} = \boxed{\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}}$

7. Given $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ show that $A^3 = I_2$ and use this to find A^{-1}. [Do not use the formula for A^{-1} to find A^{-1} here.]

First $A^3 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Next since $A^3 = I_2$, which means that $AA^2 = I_2$, we see that $A^{-1} = A^2 = \boxed{\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}}$

8. Show that $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ form a basis of the space $M_{2\times 2}(\mathbb{R})$

Proof: $\{E_1, E_2, E_3, E_4\}$ is clearly a linearly independent spanning set, hence a basis:

(1) $\lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + \lambda_4 E_4 = \vec{0} \Rightarrow \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0 \quad \therefore$ lin. indep.

(2) $aE_1 + bE_2 + cE_3 + dE_4 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \therefore$ spanning. □

9. Given $A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 5 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & -1 & 3 \\ 2 & -1 & -1 & 0 \end{pmatrix}$ find AB.

$$AB = \boxed{\begin{pmatrix} 3 & -1 & -2 & 3 \\ 0 & 1 & -1 & 6 \\ 11 & -3 & -8 & 15 \end{pmatrix}}$$

10. Find the inverse of the matrix $A = \begin{pmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{pmatrix}$

$$A^{-1} = \frac{1}{\sin^2(\theta) + \cos^2(\theta)} \begin{pmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{pmatrix}}$$