1. Show that the transformations given below are linear:

(a) $T: \mathbb{R}^2 \to \mathbb{R}^2$ with $T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}$

b) $T: \mathbb{R}^2 \to \mathbb{R}^2$ with $T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 + u_2 \\ -u_2 \end{pmatrix}$

c) $T: \mathbb{C} \to \mathbb{C}$ with $T(z) = \overline{z}$

Solution:

(b) The easiest way to do this is to realize this transformation as a matrix multiplication $T(\vec{x}) = A\vec{x}$:

$$T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 + u_2 \\ -u_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

This would give us linearity immediately since

(1) $$T(\vec{x}+\vec{y}) = A(\vec{x}+\vec{y}) = A(\vec{x}) + A(\vec{y}) = T(\vec{x}) + T(\vec{y})$$

and

(2) $$T(t\vec{x}) = A(t\vec{x}) = tA\vec{x} = tT(\vec{x})$$

Of course we can also do it the "long" way:

Pick two arbitrary vectors: $\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} A \\ B \end{pmatrix}$ then

(1) $$T(\vec{x}+\vec{y}) = T\left(\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} A \\ B \end{pmatrix}\right) = T\begin{pmatrix} a+A \\ b+B \end{pmatrix} = \begin{pmatrix} (a+A)+(b+B) \\ -(b+B) \end{pmatrix}$$

$$T(\vec{x}) + T(\vec{y}) = T\begin{pmatrix} a \\ b \end{pmatrix} + T\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a+b \\ -b \end{pmatrix} + \begin{pmatrix} A+B \\ -B \end{pmatrix} = \begin{pmatrix} (a+b)+(A+B) \\ -b-B \end{pmatrix} \quad \Rightarrow$$

$$\Rightarrow \quad T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$$

and

(2) $$T(t\vec{x}) = T\left(t\begin{pmatrix} a \\ b \end{pmatrix}\right) = T\begin{pmatrix} ta \\ tb \end{pmatrix} = \begin{pmatrix} ta+tb \\ -tb \end{pmatrix}$$

$$tT(\vec{x}) = tT\begin{pmatrix} a \\ b \end{pmatrix} = t\begin{pmatrix} a+b \\ -b \end{pmatrix} = \begin{pmatrix} t(a+b) \\ -tb \end{pmatrix} \quad \Rightarrow$$

$$\Rightarrow \quad T(t\vec{x}) = tT(\vec{x})$$

□

2. Show that the following transformations are **not** linear:

(a) $T:\ \mathbb{R}^2 \to \mathbb{R}^2$ with
$$T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2u_1 \\ 1-u_2 \end{pmatrix}$$

b) $T:\ \mathbb{R} \to \mathbb{R}^2$ with
$$T(x) = \begin{pmatrix} x \\ \sqrt{x} \end{pmatrix}$$

c) $T:\ \mathbb{C} \to \mathbb{C}$ with
$$T(z) = \frac{1}{z}$$

(d) $T: M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ with
$$T(A) = \det(A)$$

e) $T:\ \mathbb{R} \to \mathbb{R}$ with
$$T(x) = x^2$$

Solution:

(e) Fact: If T is linear than $T(\vec{0}) = \vec{0}$, or equivalently:
If $T(\vec{0}) \neq \vec{0}$ then T cannot be linear.

Unfortunately $T(0) = 0$ so this does *not* disprove linearity
[This just tells us that T might be linear.]

we can look for *one* counter example to the condition $T(\vec{x}+\vec{y}) = T(\vec{x})+T(\vec{y})$

$$\left.\begin{aligned} T(1+2) &= T(3) = 9 \\ T(1)+T(2) &= 1+4 = 5 \end{aligned}\right\} \Rightarrow \quad T(\vec{x}+\vec{y}) \neq T(\vec{x})+T(\vec{y})$$

$\therefore$ T is not linear.

Or we can look for *one* counter example to the condition $T(t\vec{x}) = tT(\vec{x})$

$$\left.\begin{aligned} T(4\cdot 4) &= T(16) = 4 \\ 4\cdot T(4) &= 4\cdot 2 = 8 \end{aligned}\right\} \Rightarrow \quad T(t\vec{x}) \neq tT(\vec{x})$$

$\therefore$ T is not linear.

4. Identify what type of linear transformations the following matrices represent:
[for example: reflection in x-axis, projection onto the line $y = x$, rotation around O over $45°$, scaling with a factor of 3, etc.]

(a) $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$ **(c)** $A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

(d) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **(e)** $A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ **(f)** $A = \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{pmatrix}$

Solutions:

(a) Reflection in the line $y = -x$ [$m = -1$ in $\frac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$]

(b) Rotation around the origin over $60°$ [$\theta = 60°$ in $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$]

(c) Scaling by a factor of 5. [Also called a dilation]

(d) The identity transformation [Every point/vector gets mapped to itself]

(e) Vertical shear with shear factor 3.

(f) Orthogonal projection onto the line $y = 2x$ [$m = 2$ in $\frac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}$]

11. Let $T(\vec{x}) = A\vec{x}$ with $A = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$.

(a) Find $T\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $T\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $T\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $T\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and $T\begin{pmatrix} 6 \\ 3 \end{pmatrix}$

(b) Find all vectors $\vec{x} \in \mathbb{R}^2$ such that $T(\vec{x}) = \vec{x}$.

(c) Find A^2.

(d) Find A^{-1}

(e) Calculate the midpoint of $\vec{x}$ and $T(\vec{x})$ and show that it lies on the line $x = 2y$.

(f) Show that T is a reflection in a line.

Solutions: (a) $T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix}$, $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7/5 \\ 1/5 \end{pmatrix}$,

$T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 11/5 \\ -2/5 \end{pmatrix}$, and $T\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

(b) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} 3x+4y=5x \\ 4x-3y=5y \end{cases} \Rightarrow \begin{cases} -2x+4y=0 \\ 4x-8y=0 \end{cases}$

$\Rightarrow x = 2y$. So all vectors of the form $\vec{x} = t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ are fixed [$T(\vec{x}) = \vec{x}$].

(c) $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **(d)** $A^{-1} = A = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$

(e) If $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ then $T(\vec{x}) = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x/5+4y/5 \\ 4x/5-3y/5 \end{pmatrix}$ hence the midpoint is

$$\frac{\vec{x}+T(\vec{x})}{2} = \frac{\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3x/5+4y/5 \\ 4x/5-3y/5 \end{pmatrix}}{2} = \begin{pmatrix} 8x/5+4y/5 \\ 4x/5+2y/5 \end{pmatrix} = \frac{(4x+2y)}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus the midpoint lies on the line $x = 2y$ which is given by $\vec{x} = t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ as we found in 11(b).

(f) From 11(b) we know that the line $x = 2y$ $(y = \frac{1}{2}x)$ is fixed, i.e. that would have to be the line of reflection. The matrix that corresponds to a reflection in the line with slope $m = \frac{1}{2}$ is $\frac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix} = \frac{1}{5/4}\begin{pmatrix} 3/4 & 1 \\ 1 & -3/4 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$

which *is* the matrix A. So indeed this transformation is a reflection in the line $y = \frac{1}{2}x$.