MAT 140 Solutions of Final Sample Questions Fall 2007

Solve the following problems.

1. (a) The eigenvalues of $A=\begin{pmatrix}1 & -1\\ 2 & -1\end{pmatrix}$ are: (10 pts)

A. $\lambda=-1$, $\lambda=1$ B. $\lambda=-1$ [✓ C.] no eigenvalues D. $\lambda=1$

[Calculating $\det(A-\lambda I)=\det\begin{pmatrix}1-\lambda & -1\\ 2 & -1-\lambda\end{pmatrix}=\lambda^2+1$. This is never zero for real λ, so there are no real eigenvalues.]

(b) True or [✓ **False**]: The vector $\vec{u}=\begin{pmatrix}1\\1\\0\end{pmatrix}$ is an eigenvector of $A=\begin{pmatrix}1 & -3 & 3\\ 3 & -3 & 3\\ 6 & -6 & 4\end{pmatrix}$.

[$A\vec{u}=\begin{pmatrix}1 & -3 & 3\\ 3 & -3 & 3\\ 6 & -6 & 4\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}-2\\0\\0\end{pmatrix}\neq\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}=\lambda\vec{u}\quad\therefore\quad A\vec{u}\neq\lambda\vec{u}$]

2. Given that $\vec{u}=\begin{pmatrix}0\\1\\-1\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}2\\0\\1\end{pmatrix}$, then: (30 pts)

(a) Find a vector $\vec{n}$ orthogonal to $\vec{u}$ and $\vec{v}$. .
(b) Find the normal / standard equation of the plane that passes through $P=(1,0,1)$ and is orthogonal to $\vec{n}$.
(c) Find the point of intersection between the plane found above and the planes $2y+z=2$, $-x+y+2z=4$, using *Gauss-Jordan Elimination*.

Solution: (a) Take $\vec{n}=\vec{u}\times\vec{v}=\begin{pmatrix}0\\1\\-1\end{pmatrix}\times\begin{pmatrix}2\\0\\1\end{pmatrix}=\boxed{\begin{pmatrix}1\\-2\\-2\end{pmatrix}}$

(b) $\vec{n}\cdot\vec{x}=\vec{n}\cdot\vec{p}$: $\begin{pmatrix}1\\-2\\-2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-2\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}\Rightarrow\boxed{x-2y-2z=-1}$

(c)

$$\left(\begin{array}{cccc}1 & -2 & -2 & -1\\ 0 & 2 & 1 & 2\\ -1 & 1 & 2 & 4\end{array}\right)\xrightarrow{R_3+R_1}\left(\begin{array}{cccc}1 & -2 & -2 & -1\\ 0 & 2 & 1 & 2\\ 0 & -1 & 0 & 3\end{array}\right)\xrightarrow{-R_3}\left(\begin{array}{cccc}1 & -2 & -2 & -1\\ 0 & 2 & 1 & 2\\ 0 & 1 & 0 & -3\end{array}\right)\xrightarrow{\text{flip } R_2\ \&\ R_3}$$

$$\begin{pmatrix} 1 & -2 & -2 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 2 & 1 & 2 \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & -2 & -2 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 8 \end{pmatrix} \xrightarrow{R_1 + 2R_3 + 2R_2} \begin{pmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 8 \end{pmatrix}$$

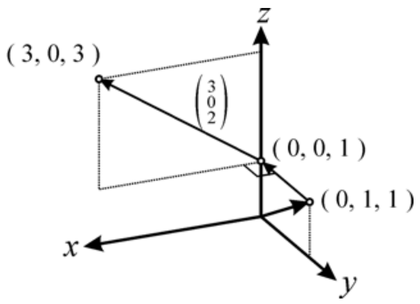
$$\therefore \boxed{(x, y, z) = (9, -3, 8)}$$

3. **(a)** Use homogeneous coordinates to find a 4×4 matrix that projects points of $\mathbb{R}^3$ on the *xz*-plane and then translates them by (3,0,2). (15 pts)

$$T(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \quad \Rightarrow \quad T\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

(b) Apply the matrix above to find the image of the point *P* = (0,1,1). Check your answer with a graph.

$$T\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \\ 1 \end{pmatrix} \quad \therefore \boxed{(0,1,1) \mapsto (3,0,3)}$$



4. Find the intersection of the two planes $x - y - 3z = 2$ and $2x - y + 2z = 5$

(a) Use Gauss-Jordan elimination.
(b) Use cross-products and the fact that the point (3, 1, 0) is on both planes. (10 pts)

Solution:

(a) $$\begin{pmatrix} 1 & -1 & -3 & 2 \\ 2 & -1 & 2 & 5 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & -1 & -3 & 2 \\ 0 & 1 & 8 & 1 \end{pmatrix} \xrightarrow{R_1 + R_2} \begin{pmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 8 & 1 \end{pmatrix}$$

Hence: $\begin{cases} x = 3 - 5z \\ y = 1 - 8z \end{cases}$ so if we take $z = t \in \mathbb{R}$ then $\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -5 \\ -8 \\ 1 \end{pmatrix}}$

(b) $\vec{n} = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ -8 \\ 1 \end{pmatrix}$ Hence $\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -5 \\ -8 \\ 1 \end{pmatrix}}$

5. (a) True or [✓ **False**] Bezier curves are not preserved under affine transformations. (15 pts)

Bezier curves ***are*** preserved under affine transformations! Affine transformations preserve ratios!

The Bezier curve determined by A, B, C and D is

$$z(t) = (1-t)^3 A + 3(1-t)^2 t\, B + 3(1-t)t^2 C + t^3 D \quad \text{for} \quad 0 \le t \le 1,$$

or in vector notation:

$$\vec{z}(t) = (1-t)^3 \vec{a} + 3(1-t)^2 t\, \vec{b} + 3(1-t)t^2 \vec{c} + t^3 \vec{d}$$

Let T be the affine map defined by $T(\vec{x}) = M\vec{x} + \vec{m}$ which maps $\vec{a},\ \vec{b},\ \vec{c}$ and $\vec{d}$ to $\vec{a}',\ \vec{b}',\ \vec{c}'$ and $\vec{d}'$ (i.e. A, B, C and D to A', B', C' and D') then

$$\begin{aligned} T(\vec{z}(t)) &= T\left((1-t)^3 \vec{a} + 3(1-t)^2 t\, \vec{b} + 3(1-t)t^2 \vec{c} + t^3 \vec{d}\right) \\ &= M\left((1-t)^3 \vec{a} + 3(1-t)^2 t\, \vec{b} + 3(1-t)t^2 \vec{c} + t^3 \vec{d}\right) + \vec{m} \\ &= (1-t)^3 M(\vec{a}) + 3(1-t)^2 t\, M(\vec{b}) + 3(1-t)t^2 M(\vec{c}) + t^3 M(\vec{d}) + \vec{m} \end{aligned}$$

Note that $(1-t)^3 + 3(1-t)^2 t + 3(1-t)t^2 + t^3 = 1$ hence

$$(1-t)^3 \vec{m} + 3(1-t)^2 t\, \vec{m} + 3(1-t)t^2 \vec{m} + t^3 \vec{m} = \vec{m}$$

which implies that

$$\begin{aligned} T(\vec{z}(t)) &= (1-t)^3 \left(M(\vec{a}) + \vec{m}\right) + 3(1-t)^2 t \left(M(\vec{b}) + \vec{m}\right) \\ &\qquad + 3(1-t)t^2 \left(M(\vec{c}) + \vec{m}\right) + t^3 \left(M(\vec{d}) + \vec{m}\right) \\ &= (1-t)^3 \vec{a}' + 3(1-t)^2 t\, \vec{b}' + 3(1-t)t^2 \vec{c}' + t^3 \vec{d}' \end{aligned}$$

and this is exactly the Bezier curve determined by A', B', C' and D' with corresponding t values. Hence the affine map T maps each point of the Bezier curve determined by A, B, C and D to the corresponding point on the Bezier curve determined by A', B', C' and D'.

(b) Find the point on the Bezier curve determined by $A = (-2,2)$, $B = (1,4)$, and $C = (3,2)$ that corresponds to $t = 1/4$.

$$z\left(\frac{1}{4}\right) = \left(\frac{3}{4}\right)^2 (-2,2) + 2\,\frac{3}{4}\,\frac{1}{4}(1,4) + \left(\frac{1}{4}\right)^2 (3,2) = \boxed{\left(\frac{-9}{16}, \frac{11}{4}\right)}$$

6. Consider the triangle $\triangle ABC$ with vertices A = (0,0), B = (2,0), and C = (1,1). (20 pts)

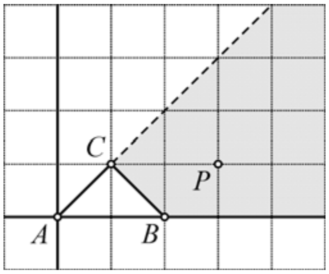
(a) What are the Barycentric Coordinates of the point A?

(1, 0, 0) since $A = 1\cdot A + 0\cdot B + 0\cdot C$

(b) Does the point P with Barycentric Coordinates P = (-1,1,1) lie in the triangle $\triangle ABC$?
Find the coordinates of P, and verify with a graph that, indeed, P does not lie in $\triangle ABC$.

P lies outside the triangle. (Only points with *all* Barycentric coordinates *positive* lie inside the triangle)

$$P = -1\cdot(0,0) + 1\cdot(2,0) + 1\cdot(1,1) = (3,1)$$



Some extra practice questions

7. Let P = (1, 2), Q = (5, 3) and R = (2, 5). Find the Barycentric coordinates of the point $X = (2, 3)$

(a) by solving a system of linear equations,
(b) by using matrices,
(c) by calculating areas.
(d) Let he point Y be given by $Y = aP + bQ + cR$ where $a + b + c = 1$, and $a > 0,\ b < 0$ and $c > 0$, indicate in what the region of the plane Y is located.

Solution:

(a)

$$\left.\begin{matrix} a\begin{pmatrix}1\\2\end{pmatrix} + b\begin{pmatrix}5\\3\end{pmatrix} + c\begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix} \\ a+b+c=1 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} a+5b+2c=2 \\ 2a+3b+5c=3 \\ a+\ b+\ c=1 \end{matrix}\right.$$

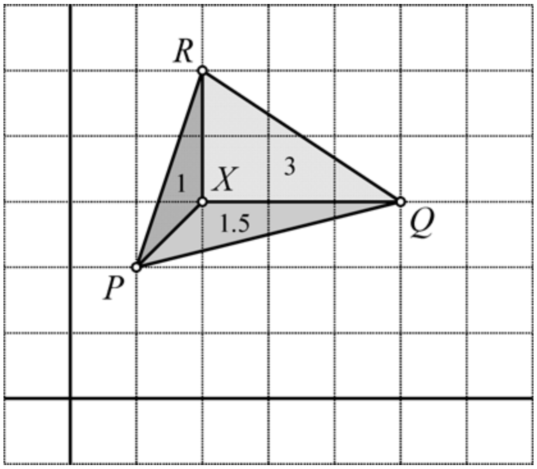
$$\begin{pmatrix} 1 & 5 & 2 & 2 \\ 2 & 3 & 5 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{Gauss Jordan}} \begin{pmatrix} 1 & 0 & 0 & 6/11 \\ 0 & 1 & 0 & 2/11 \\ 0 & 0 & 1 & 3/11 \end{pmatrix} \quad \therefore \left\{\begin{matrix} a = 6/11 \\ b = 2/11 \\ c = 3/11 \end{matrix}\right.$$

(b)

$$\begin{pmatrix} 1 & 5 & 2 \\ 2 & 3 & 5 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 5 & 2 \\ 2 & 3 & 5 \\ 1 & 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -2 & -3 & 19 \\ 3 & -1 & -1 \\ -1 & 4 & -7 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6/11 \\ 2/11 \\ 3/11 \end{pmatrix}$$

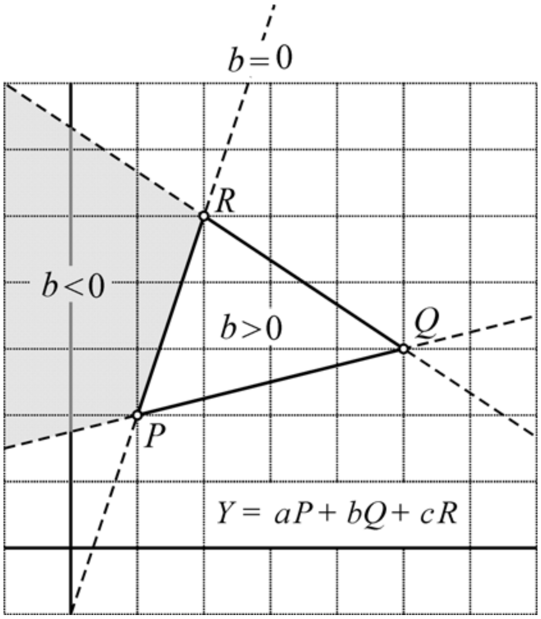
(c)



$$\text{Area}(\triangle XQR) = 3$$
$$\text{Area}(\triangle PXR) = 1$$
$$\text{Area}(\triangle PQX) = 1.5$$
$$\text{Area}(\triangle PQR) = 3 + 1 + 1.5 = 5.5$$

$$a = \frac{\text{Area}(\triangle XQR)}{\text{Area}(\triangle PQR)} = \frac{3}{5.5} = \frac{6}{11}$$
$$b = \frac{\text{Area}(\triangle PXR)}{\text{Area}(\triangle PQR)} = \frac{1}{5.5} = \frac{2}{11}$$
$$c = \frac{\text{Area}(\triangle PQX)}{\text{Area}(\triangle PQR)} = \frac{1.5}{5.5} = \frac{3}{11}$$

(d)



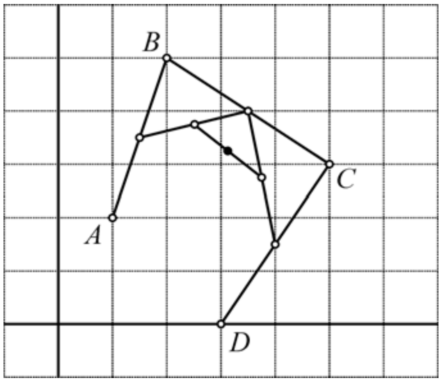
8. Let $A = (1, 2)$, $B = (2, 5)$, $C = (5, 3)$ and $D = (3, 0)$. Let $z(t)$ be the Bezier curve determined by A, B, C and D (in the given order).

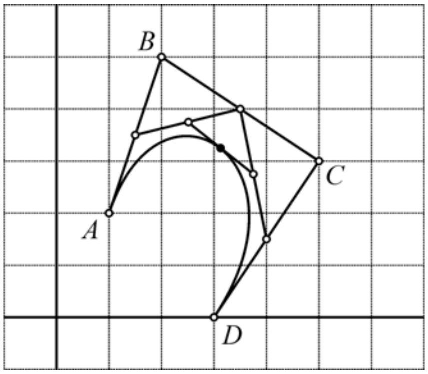
(a) Sketch the point on the Bezier curve $z(t)$ at $t = \frac{1}{2}$.

(b) What points are determined by $z(0)$ and $z(1)$?

(c) Sketch the entire Bezier curve $z(t)$ for $0 \leq t \leq 1$.

Solution: (a) ↓ **(b)** $z(0)=A$ and $z(1)=D$ **(c)** ↓





9. A light source is located at point $L=(10,5,7)$. Locate the shadow of the point $A=(2,3,5)$ on the plane $x-y+2z=1$.

Solution: Intersect $\begin{cases} x=10+t(10-2)=10+8t \\ y= \ 5+t(5-3) \ = \ 5+2t \\ z= \ 7+t(7-5) \ = \ 7+2t \end{cases}$ with plane $x-y+2z=1$:

$$10+8t-(5+2t)+2(7+2t)=1 \quad \Rightarrow \quad t=-\frac{9}{5} \quad \Rightarrow \quad \boxed{A'=\left(-\frac{22}{5},\frac{7}{5},\frac{17}{5}\right)}$$

10. A reflection in the plane $x-y=3$ is followed by a translation over $\begin{pmatrix}1\\2\\3\end{pmatrix}$.

Use homogeneous coordinates to describe the transformation as

$$T\begin{pmatrix}x\\y\\z\\1\end{pmatrix}=\begin{pmatrix}* & * & * & *\\ * & * & * & *\\ * & * & * & *\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\\1\end{pmatrix}$$

Solution: The reflection in the plane (which does NOT go to the origin) is achieved as follows:

$$R\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&1&0\\1&1&0\\0&0&2\end{pmatrix}\left(\begin{pmatrix}x\\y\\z\end{pmatrix}-\begin{pmatrix}3\\0\\0\end{pmatrix}\right)+\begin{pmatrix}3\\0\\0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&1&0\\1&1&0\\0&0&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}+\frac{1}{2}\begin{pmatrix}3\\-3\\0\end{pmatrix}$$ hence the entire map is

$$\boxed{T\begin{pmatrix}x\\y\\z\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&0&\frac{5}{2}\\ \frac{1}{2}&\frac{1}{2}&0&\frac{1}{2}\\ 0&0&1&3\\ 0&0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\\1\end{pmatrix}}$$

11. Find the intersection of the two planes $2x+y+5z=2$ and $x+y-z=3$

(a) Use Gauss-Jordan elimination.
(b) Use cross-products and the fact that the point $(-1, 4, 0)$ is on both planes.

Solution: (a)

$$\begin{pmatrix} 2 & 1 & 5 & 2 \\ 1 & 1 & -1 & 3 \end{pmatrix} \xrightarrow{\text{swap rows}} \begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & 1 & 5 & 2 \end{pmatrix} \xrightarrow{\mathrm{R}_2-2\mathrm{R}_1} \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -1 & 7 & -4 \end{pmatrix} \longrightarrow$$

$$\xrightarrow{\mathrm{R}_1+\mathrm{R}_2} \begin{pmatrix} 1 & 0 & 6 & -1 \\ 0 & -1 & 7 & -4 \end{pmatrix} \xrightarrow{-1\times\mathrm{R}_2} \begin{pmatrix} 1 & 0 & 6 & -1 \\ 0 & 1 & -7 & 4 \end{pmatrix} \text{ Hence } \begin{cases} x=-1-6z \\ y=\ \ 4+7z \end{cases}$$

which gives us (when we take $z=t$) the line: $\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} + t\begin{pmatrix} -6 \\ 7 \\ 1 \end{pmatrix}}$.

(b) The cross-product of the normals of the two planes is a perfect direction vector of the line of intersection the planes: $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 7 \\ 1 \end{pmatrix}$. Hence with the given point $(-1, 4, 0)$ used as position vector we get the same equation: $\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} + t\begin{pmatrix} -6 \\ 7 \\ 1 \end{pmatrix}}$.

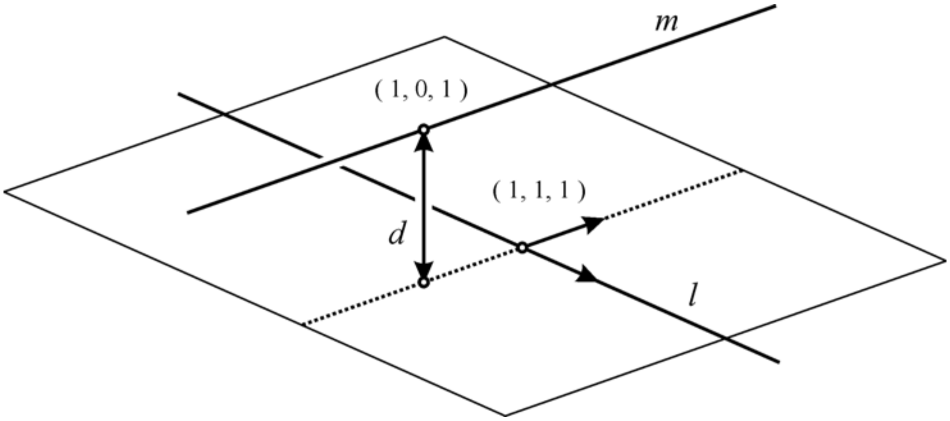
12. Find the distance between the lines $l: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $m: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

Solution:

Note that when we ‘add’ the direction vector of the second line to the first line we get the equation of the plane:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

This plane is parallel to both lines! [One lies in it, the other doesn't but is parallel to it.]



A normal for this plane is $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ 5 \end{pmatrix}$ hence its normal equation: $x - 7y - 5z = -11$

i.e. $x - 7y - 5z + 11 = 0$ and hence the distance from the point (1, 0, 1) on m to the plane is given by

$$d = \frac{|1 - 7\cdot 0 - 5\cdot 1 + 11|}{\sqrt{1^2 + 7^2 + 5^2}} = \boxed{\frac{7\sqrt{3}}{15}}$$