Coordinates, bases and change of basis matrices.

1. Let $\beta = \{\vec{\beta}_1, \vec{\beta}_2\}$ be an ordered basis with $\vec{\beta}_1 = \begin{pmatrix}3\\1\end{pmatrix}$ and $\vec{\beta}_2 = \begin{pmatrix}5\\2\end{pmatrix}$, and

$\gamma = \{\vec{\gamma}_1, \vec{\gamma}_2\}$ the standard, ordered basis with $\vec{\gamma}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\vec{\gamma}_2 = \begin{pmatrix}0\\1\end{pmatrix}$.

(a) Find $\begin{pmatrix}2\\-1\end{pmatrix}_\beta$, $\begin{pmatrix}1\\-1\end{pmatrix}_\beta$, $\begin{pmatrix}1\\0\end{pmatrix}_\beta$, $\begin{pmatrix}0\\1\end{pmatrix}_\beta$ and $\begin{pmatrix}2\\-1\end{pmatrix}_\gamma$. Find $\begin{pmatrix}1\\-1\end{pmatrix}_\beta + \begin{pmatrix}2\\3\end{pmatrix}_\gamma$ (Careful !)

(b) Express $\vec{\gamma}_1$ in terms of basis $\beta = \{\vec{\beta}_1, \vec{\beta}_2\}$

[i.e. $\vec{\gamma}_1 = \begin{pmatrix}1\\0\end{pmatrix} = a\vec{\beta}_1 + b\vec{\beta}_2 = a\begin{pmatrix}3\\1\end{pmatrix} + b\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}_\beta$]

(c) Express $\vec{\gamma}_2$ in terms of basis $\beta = \{\vec{\beta}_1, \vec{\beta}_2\}$.

[i.e. $\vec{\gamma}_2 = \begin{pmatrix}0\\1\end{pmatrix} = c\vec{\beta}_1 + d\vec{\beta}_2 = c\begin{pmatrix}3\\1\end{pmatrix} + d\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}c\\d\end{pmatrix}_\beta$]

(d) Find the change of basis matrix M that converts a vector $\vec{x} = \begin{pmatrix}a\\b\end{pmatrix}_\beta$ expressed with respect to basis $\beta = \{\vec{\beta}_1, \vec{\beta}_2\}$ to the same vector $\vec{x} = \begin{pmatrix}A\\B\end{pmatrix}_\gamma$ but now expressed with respect to the basis $\gamma = \{\vec{\gamma}_1, \vec{\gamma}_2\}$: i.e. $M\begin{pmatrix}a\\b\end{pmatrix}_\beta = \begin{pmatrix}A\\B\end{pmatrix}_\gamma$.

(e) Find the change of basis matrix N that converts a vector $\vec{x} = \begin{pmatrix}A\\B\end{pmatrix}_\gamma$ with respect to γ to the same vector $\vec{x} = \begin{pmatrix}a\\b\end{pmatrix}_\beta$ but now with respect to β i.e. $N\begin{pmatrix}A\\B\end{pmatrix}_\gamma = \begin{pmatrix}a\\b\end{pmatrix}_\beta$.

(How are M and N related?)

Solution: (a) $\begin{pmatrix}2\\-1\end{pmatrix}_\beta = 2\begin{pmatrix}3\\1\end{pmatrix} - 1\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$; $\begin{pmatrix}1\\-1\end{pmatrix}_\beta = 1\begin{pmatrix}3\\1\end{pmatrix} - 1\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}-2\\-1\end{pmatrix}$;

$\begin{pmatrix}1\\0\end{pmatrix}_\beta = 1\begin{pmatrix}3\\1\end{pmatrix} + 0\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix} = \vec{\beta}_1$; $\begin{pmatrix}0\\1\end{pmatrix}_\beta = 0\begin{pmatrix}3\\1\end{pmatrix} + 1\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix} = \vec{\beta}_2$;

$\begin{pmatrix}2\\-1\end{pmatrix}_\gamma = 2\begin{pmatrix}1\\0\end{pmatrix} - 1\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}_\beta + \begin{pmatrix}2\\3\end{pmatrix}_\gamma = 1\begin{pmatrix}3\\1\end{pmatrix} - 1\begin{pmatrix}5\\2\end{pmatrix} + \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$

(b) When $a\begin{pmatrix}3\\1\end{pmatrix}+b\begin{pmatrix}5\\2\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$ i.e. $\begin{pmatrix}3&5\\1&2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}_\beta=\begin{pmatrix}1\\0\end{pmatrix}$ then $\begin{pmatrix}a\\b\end{pmatrix}_\beta=\begin{pmatrix}3&5\\1&2\end{pmatrix}^{-1}\begin{pmatrix}1\\0\end{pmatrix}=$

$=\begin{pmatrix}2&-5\\-1&3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\-1\end{pmatrix}_\beta$ In (a) we checked that indeed $\begin{pmatrix}2\\-1\end{pmatrix}_\beta=\begin{pmatrix}1\\0\end{pmatrix}=\vec{\gamma}_1$.

(c) When $a\begin{pmatrix}3\\1\end{pmatrix}+b\begin{pmatrix}5\\2\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$ i.e. $\begin{pmatrix}3&5\\1&2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$ then $\begin{pmatrix}a\\b\end{pmatrix}_\beta=\begin{pmatrix}3&5\\1&2\end{pmatrix}^{-1}\begin{pmatrix}0\\1\end{pmatrix}=$

$=\begin{pmatrix}2&-5\\-1&3\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-5\\3\end{pmatrix}_\beta$. Check that indeed $\begin{pmatrix}-5\\3\end{pmatrix}_\beta=\begin{pmatrix}0\\1\end{pmatrix}=\vec{\gamma}_2$.

(d) When $\begin{pmatrix}a\\b\end{pmatrix}_\beta=a\begin{pmatrix}3\\1\end{pmatrix}+b\begin{pmatrix}5\\2\end{pmatrix}=\begin{pmatrix}A\\B\end{pmatrix}_\gamma$ i.e. $\begin{pmatrix}3&5\\1&2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}_\beta=\begin{pmatrix}A\\B\end{pmatrix}_\gamma$ hence the required matrix is $M=\begin{pmatrix}3&5\\1&2\end{pmatrix}$

(e) From (d) we see that $\begin{pmatrix}a\\b\end{pmatrix}_\beta=\begin{pmatrix}3&5\\1&2\end{pmatrix}^{-1}\begin{pmatrix}A\\B\end{pmatrix}_\gamma=\begin{pmatrix}2&-5\\-1&3\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}_\gamma$ so that

$N=\begin{pmatrix}2&-5\\-1&3\end{pmatrix}=M^{-1}$

5. (a) Find the eigenvalues of $T(\vec{x})=B\vec{x}$ where $B=\begin{pmatrix}3&6\\9&6\end{pmatrix}$

(b) For each eigenvalue find the corresponding eigenvectors.

(c) Pick a basis of eigenvectors [Call it $\gamma=\{\vec{\gamma}_1,\vec{\gamma}_2\}$]

Solution:

(a) $\det\begin{pmatrix}3-\lambda&6\\9&6-\lambda\end{pmatrix}=(\lambda-12)(\lambda+3)$ hence the eigenvalues are $\lambda=12,-3$.

(b) When $\lambda=12$ then $\begin{pmatrix}-9&6\\9&-6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ implies $3x=2y$ $\therefore$ $\begin{pmatrix}x\\y\end{pmatrix}=t\begin{pmatrix}2\\3\end{pmatrix}$

When $\lambda=-3$ then $\begin{pmatrix}6&6\\9&9\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ implies $y=-x$ $\therefore$ $\begin{pmatrix}x\\y\end{pmatrix}=t\begin{pmatrix}1\\-1\end{pmatrix}$

(c) We could pick $\gamma=\{\vec{\gamma}_1,\vec{\gamma}_2\}$ with $\vec{\gamma}_1=\begin{pmatrix}2\\3\end{pmatrix}$ and $\vec{\gamma}_2=\begin{pmatrix}1\\-1\end{pmatrix}$

6. **(a)** Find the eigenvalues of $T(\vec{x}) = C\vec{x}$ where $C = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}$

(b) For each eigenvalue find the corresponding eigenvectors.

(c) Does there exist a basis of eigenvectors [If so call it $\delta = \left\{ \vec{\delta}_1, \vec{\delta}_2 \right\}$]

Solution: **(a)** $\det\begin{pmatrix} 2-\lambda & 0 \\ 3 & 2-\lambda \end{pmatrix} = (\lambda-2)^2$ hence the only eigenvalue of T is 2.

(b) When $\lambda = 2$ then $\begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ implies $\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(c) There are no two linearly independent eigenvectors for T. [The only eigenvectors of T are all multiples of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$] hence there is no basis of eigenvectors.

10. Find a parametric equation of the line through the point $(2, 4, 5)$ and perpendicular to the plane $3x + 7y - 5z = 21$

Solution: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ 7 \\ -5 \end{pmatrix}$ or equivalently $\begin{cases} x = 2 + 3t \\ y = 4 + 7t \\ z = 5 - 5t \end{cases}$

12. A plane goes through the points $(2, 4, 5)$, $(1, 5, 7)$ and $(-1, 6, 8)$.

(a) Find an equation $ax + by + cz = d$ for the plane.

(b) Find a vector or parametric equation of the plane.

Solution:

(a) $\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix}$ are direction vectors for the plane, hence a normal is found by cross-product $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$ so that the plane looks like $-x - 3y + z = d$.

To find d substitute one of the points: $-2 - 3 \cdot 4 + 5 = d = -9$

Hence:

$$x + 3y - z = 9$$

or

Solve $\begin{cases} 2a+4b+5c=1 \\ a+5b+7c=1 \\ -a+6b+8c=1 \end{cases}$ i.e. $\begin{pmatrix} 2 & 4 & 5 \\ 1 & 5 & 7 \\ -1 & 6 & 8 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 & 4 & 5 \\ 1 & 5 & 7 \\ -1 & 6 & 8 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

which gives us: $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 & 4 & 5 \\ 1 & 5 & 7 \\ -1 & 6 & 8 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/9 & 2/9 & -1/3 \\ 5/3 & -7/3 & 1 \\ -11/9 & 16/9 & -2/3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/9 \\ 3/9 \\ -1/9 \end{pmatrix}$

so that $\frac{1}{9}x+\frac{3}{9}y-\frac{1}{9}z=1$ i.e. $x+3y-z=9$

(c) There are many possible answers, and many ways to do it:

(1) Note $z = x+3y-9$ i.e. $\begin{cases} x = x \\ y = \quad y \\ z = x+3y-9 \end{cases}$ i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\cdot\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y\cdot\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -9 \end{pmatrix}$

or

(2) we can also use the direction vectors we found in (a)

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} + t\cdot\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + s\cdot\begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix}$ i.e. $\begin{cases} x = 2+t+3s \\ y = 4-t-2s \\ z = 5-2t-3s \end{cases}$

13. (a) Find the intersection of the two lines given parametrically by

$l: \begin{cases} x = 2t+1 \\ y = 3t+2 \\ z = 4t+3 \end{cases}$ and $m: \begin{cases} x = s+2 \\ y = 2s+4 \\ z = -4s-1 \end{cases}$

(b) Find the equation $ax+by+cz=d$ of plane determined by these two lines.

Solution:

(a) $\begin{cases} 2t+1 = s+2 \\ 3t+2 = 2s+4 \\ 4t+3 = -4s-1 \end{cases} \Rightarrow \begin{cases} 2t-s=1 \\ 3t-2s=2 \\ t+s=-1 \end{cases} \Rightarrow \begin{cases} t=0 \\ s=-1 \end{cases}$ hence the point of

intersection is $\begin{cases} x = 2\cdot 0+1=1 \\ y = 3\cdot 0+2=2 \\ z = 4\cdot 0+3=3 \end{cases}$ i.e. $(1,2,3)$

(b) The direction vectors of the lines $\begin{pmatrix}2\\3\\4\end{pmatrix}$ and $\begin{pmatrix}1\\2\\-4\end{pmatrix}$ are also direction vectors of the plane through these lines. Hence a normal for the plane is $\begin{pmatrix}2\\3\\4\end{pmatrix}\times\begin{pmatrix}1\\2\\-4\end{pmatrix}=\begin{pmatrix}-20\\12\\1\end{pmatrix}$

hence: $-20x+12y+z=d$ and since $(1,2,3)$ is on the plane we find

$$-20x+12y+z=7$$

15. Find the length and direction of the vectors $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$ when

(a) $\vec{u}=\begin{pmatrix}2\\3\\0\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}-1\\1\\0\end{pmatrix}$ **(b)** $\vec{u}=\begin{pmatrix}1\\3\\-2\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}2\\4\\5\end{pmatrix}$

(c) $\vec{u}=\begin{pmatrix}2\\0\\0\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}0\\-3\\0\end{pmatrix}$

Solution:

(a) $\vec{u}\times\vec{v}=\begin{pmatrix}2\\3\\0\end{pmatrix}\times\begin{pmatrix}-1\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\5\end{pmatrix}$ and $\vec{v}\times\vec{u}=\begin{pmatrix}-1\\1\\0\end{pmatrix}\times\begin{pmatrix}2\\3\\0\end{pmatrix}=\begin{pmatrix}0\\0\\-5\end{pmatrix}$

So $\vec{u}\times\vec{v}$ points in the direction of the positive z-axis and has length 5, and $\vec{v}\times\vec{u}$ points in the direction of the negative z-axis and has length 5 too.

(b) $\vec{u}\times\vec{v}=\begin{pmatrix}1\\3\\-2\end{pmatrix}\times\begin{pmatrix}2\\4\\5\end{pmatrix}=\begin{pmatrix}23\\-9\\-2\end{pmatrix}$ and $\vec{v}\times\vec{u}=\begin{pmatrix}2\\4\\5\end{pmatrix}\times\begin{pmatrix}1\\3\\-2\end{pmatrix}=\begin{pmatrix}-23\\9\\2\end{pmatrix}$

So $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$ point in the indicated directions (the are opposites of each other) and both have length $\sqrt{614}$

(c) $\vec{u}\times\vec{v}=\begin{pmatrix}2\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\-3\\0\end{pmatrix}=\begin{pmatrix}0\\0\\-6\end{pmatrix}$ and $\vec{v}\times\vec{u}=\begin{pmatrix}0\\-3\\0\end{pmatrix}\times\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\6\end{pmatrix}$

So $\vec{u} \times \vec{v}$ points in the direction of the negative z-axis and has length 6, and $\vec{v} \times \vec{u}$ points in the direction of the posative z-axis and has length 5 too.

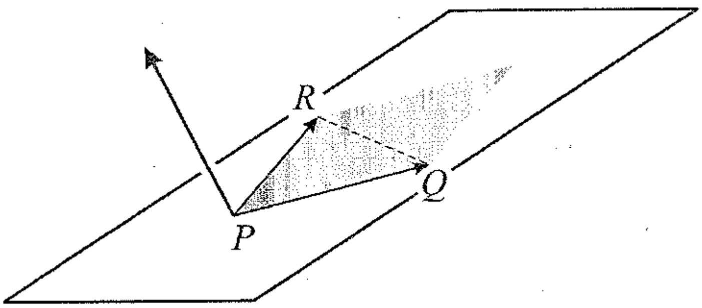
16. Let $P = (1,-1,2)$, $Q = (2,0,-1)$ and $\mathrm{R} = (0,2,1)$

(a) Find the area of the triangle determined by P, Q and R.
(b) Find the unit vector perpendicular to the plane PQR.

Solution:

(a) $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$ and $\overrightarrow{PR} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$ span a parallelogram that is twice the size of the triangle and the cross-product gives us the area of the parallelogram, hence:

$$\text{Area}(\Delta PQR) = \frac{1}{2}\left\| \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} \right\| = \frac{1}{2}\left\| \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix} \right\| = \frac{1}{2}\sqrt{96} = 2\sqrt{6}$$



(b) $\begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix}$ is perpendicular to the plane, but is not a unit vector: hence we can take

$$\frac{1}{\sqrt{96}}\begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} \sqrt{6}/3 \\ \sqrt{6}/6 \\ \sqrt{6}/6 \end{pmatrix}$$