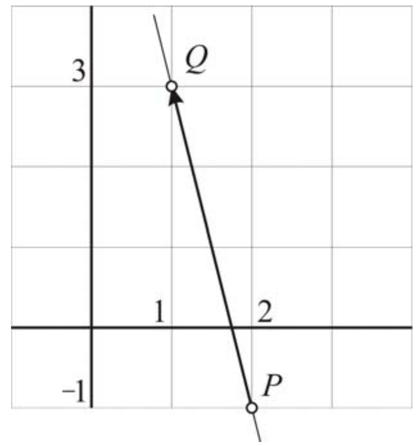
MAT 140 **Test 1 Solutions** **Fall 2007**

1. (a) Find the distance between $P = (2, -1)$ and $Q = (1, 3)$.

$$\text{dist}(P,Q) = \sqrt{(2-1)^2 + (-1-3)^2} = \boxed{\sqrt{17}}$$



(b) Find the midpoint of the line segment joining the points P and Q.

$$M = \left(\frac{2+1}{2}, \frac{-1+3}{2}\right) = \boxed{\left(\frac{3}{2}, 1\right)}$$

(c) Find the equation of the line passing through the points P and Q.

The slope of the line is -4. Using the point-slope form with $P = (2, -1)$ we get

$$y - 3 = -4(x-1) \quad \Rightarrow \quad \boxed{y = -4x + 7}$$ [or equivalently $4x + y = 7$] (15 pts)

2. Let $\vec{u} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Find:

(a) $2(\vec{u} + \vec{w}) - 3\vec{v} = 2\left(\begin{pmatrix} -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right) - 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} -1 \\ 4 \end{pmatrix}}$

(b) $\|\vec{w}\| = \sqrt{2^2 + (-1)^2} = \boxed{\sqrt{5}}$ (10 pts)

3. The vector $\vec{u}$ has initial point $P = (2, -3)$ and terminal point $Q = (-1, 2)$.

(a) Find its position vector: $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \boxed{\begin{pmatrix} -3 \\ 5 \end{pmatrix}}$

(b) Find the unit vector having the same direction as $\vec{u}$.

Since $\|\overrightarrow{PQ}\| = \sqrt{9+25} = \sqrt{34}$ the requested vector is $\boxed{\frac{1}{\sqrt{34}}\begin{pmatrix} -3 \\ 5 \end{pmatrix}}$

(c) Write $\vec{u}$ in the form $\vec{u} = a\vec{\varepsilon}_1 + b\vec{\varepsilon}_2$, where $\vec{\varepsilon}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{\varepsilon}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$\boxed{\begin{pmatrix} -3 \\ 5 \end{pmatrix} = -8\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

(15 pts)

4. (a) Find the angle between the vectors $\vec{u} = \begin{pmatrix} 2 \\ \sqrt{12} \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} \sqrt{2}/4 \\ 0 \end{pmatrix}$.

$$\cos(\theta) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{2\cdot\frac{\sqrt{2}}{4} - \sqrt{12}\cdot 0}{\sqrt{4+12}\,\sqrt{\frac{2}{16}+0}} = \frac{1}{2} \quad \text{hence} \quad \boxed{\theta = \frac{\pi}{3} \quad [\text{ or } \ \theta = 60^\circ\ ]}$$

(b) Find the orthogonal projection of $\vec{v}$ onto $\vec{u}$.

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = \frac{\sqrt{2}/2}{16}\begin{pmatrix} 2 \\ \sqrt{12} \end{pmatrix} = \boxed{\frac{\sqrt{2}}{32}\begin{pmatrix} 2 \\ \sqrt{12} \end{pmatrix}} = \frac{\sqrt{2}}{16}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$$

(10 pts)

5. (a) Show that the vectors $\vec{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ are linearly dependent.

Note $\vec{u} + 2\vec{v} - \vec{w} = \vec{0}$ hence they are linearly dependent.

(b) Show that the vectors $\vec{u}$ and $\vec{v}$ above are linearly independent.

$$a\begin{pmatrix} -1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} -a + 2b = 0 \\ \ \ a + b \ = 0 \end{cases} \quad \Rightarrow \quad 3b = 0 \quad \therefore \quad b = 0 \ \text{ and } \ a = 0$$

Hence the vectors $\vec{u}$ and $\vec{v}$ are linearly independent.

(c) Show that $\mathbb{R}^2$ is spanned by the vectors $\vec{u}$ and $\vec{v}$ above.

$$a\begin{pmatrix} -1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad \begin{cases} -a + 2b = x \\ \ \ a + \ b = y \end{cases} \quad \Rightarrow \quad 3b = x + y \quad \therefore \quad b = \frac{x+y}{3}$$

$$\text{and} \quad a = \frac{2y - x}{3}$$

Hence the vectors $\vec{u}$ and $\vec{v}$ span $\mathbb{R}^2$**.**

(d) (i) Do the vectors $\vec{u}$ and $\vec{v}$ above form a basis for $\mathbb{R}^2$?
Yes, they form a linearly independent spanning set [by 5(b) and (c)].

(ii) Do they form an orthogonal basis ? **No**, since $\vec{u}\cdot\vec{v} \neq 0$.

(iii) What is the dimension of $\mathbb{R}^2$? $\dim(\mathbb{R}^2) = \mathbf{2}$.

(30 pts)

6. (a) Exhibit a basis for the vector space $M_{2\times 2}(\mathbb{R})$, other than the "natural" basis.

Lots of answers are possible, e.g. $\boxed{\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}}$

(b) The dimension of $M_{2\times 2}(\mathbb{R})$ is: A. 2 B. 3 C. $\boxed{\mathbf{4}}$ D. 8 .

(c) $\boxed{\textbf{True}}$ or **False**: The following vectors can *not* be a basis for $P_2[x]$:

$$p(x) = 1 + x \ , \ q(x) = 2 + 2x \ , \ r(x) = x + x^2$$

Since $q(x) = 2p(x)$ these vectors are dependent, hence they cannot form a basis.

(e) $\boxed{\textbf{True}}$ or **False**: The dimension of $P_2[x]$ is 3.

Since $\{1, x, x^2\}$ is a basis of $P_2[x]$ we have $\dim(P_2[x]) = 3$. (20 pts)