Key

TEST 2

(Math 140-A)

Solve the following problems. Show all your work in the space under each problem.

1. (a) **True** or **False**: The vector $\vec{u} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$.

True, bec. $\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix} = 3\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ multiple of $\vec{u}$ //

(b) Find the eigenvalues of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$. (15 pts)

$\det(A - \lambda I) = 0 \Rightarrow \det\begin{pmatrix} 2-\lambda & -4 \\ -1 & -1-\lambda \end{pmatrix} = 0 \Rightarrow (2-\lambda)(-1-\lambda) - 4 = 0$

$\Rightarrow \lambda^2 - \lambda - 6 = 0 \Rightarrow (\lambda-3)(\lambda+2) = 0 < \boxed{\lambda_1 = 3}, \boxed{\lambda_2 = -2}$ //

(c) Find the corresponding eigenvectors for each eigenvalue of A.

For $\lambda_1 = 3$: $\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{Bmatrix} 2x_1 - 4x_2 = 3x_1 \\ -x_1 - x_2 = 3x_2 \end{Bmatrix} \rightarrow x_1 = -4x_2$

So, $\vec{u}_1 = \begin{pmatrix} -4x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -4 \\ 1 \end{pmatrix} \quad \left(= \begin{pmatrix} -4 \\ 1 \end{pmatrix}\right)$

For $\lambda_2 = -2$: $\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{Bmatrix} 2x_1 - 4x_2 = -2x_1 \\ -x_1 - x_2 = -2x_2 \end{Bmatrix} \rightarrow x_1 = x_2$

So, $\vec{u}_2 = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \left(= \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$ //

2. Given that $\vec{u} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$, then: (15 pts)

(a) Find the cross product of $\vec{u}$ and $\vec{v}$.

$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 0 & -1 & 1 \\ 2 & 3 & 0 \end{vmatrix} = \vec{e}_1(-3) - \vec{e}_2(-2) + \vec{e}_3(2) = \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$ //

(b) **True** or **False**: $\vec{u}$ and $\vec{v}$ are parallel.

False, bec. $\vec{u} \times \vec{v} \neq \vec{0}$ //

(c) Find the area of the parallelogram generated by $\vec{u}$ and $\vec{v}$.

Area $= \|\vec{u} \times \vec{v}\| = \sqrt{(-3)^2 + (-2)^2 + 2^2} = \sqrt{17}$ //

3. Given the plane M: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix}$ and the line l: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} -2 \\ -6 \\ -2 \end{pmatrix}$:

(a) Express the plane M in the form $ax + by + cz = d$. (10 pts)

From M, we have : $\vec{u} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix}$, and $P = (-1, 1, 2)$.

So, $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 3 & 0 \\ 2 & 7 & 1 \end{vmatrix} = \vec{e}_1(3) - \vec{e}_2(1) + \vec{e}_3(1) = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$.

Hence, M : $3(x+1) + (-1)(y-1) + 1(z-2) = 0 \Rightarrow 3x - y + z = -2$. //

(b) Find the point of intersection of the plane M and the line l.

From l, we have ; $x = 2-2t$, $y = 2-6t$, $z = -1-2t$.

Putting l in M ; $3(2-2t) - (2-6t) + (-1-2t) = -2$

$\Rightarrow 6 - \cancel{6t} - 2 + \cancel{6t} - 1 - 2t = -2 \longrightarrow t = 5/2$.

So, for $t = 5/2$ in l, we get : $Q = (-3, -13, -6)$. //

4. Given $A = \begin{pmatrix} 2 & 1 & 0 \\ -2 & 0 & -2 \\ 3 & 2 & 1 \end{pmatrix}$, find : (15 pts)

(a) The det(A).

$\det(A) = 2(4) - 1(4) + 0(-4) = 4$ //

(b) Find the cofactor matrix of A.

$$C_A = \begin{pmatrix} \begin{vmatrix} 0 & -2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & -2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} -2 & 0 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ -2 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 4 & -4 & -4 \\ -1 & 2 & -1 \\ -2 & 4 & 2 \end{pmatrix}$$ //

(c) Find A^{-1} using the adjoint of A.

$$\tilde{A} = C_A^t = \begin{pmatrix} 4 & -1 & -2 \\ -4 & 2 & 4 \\ -4 & -1 & 2 \end{pmatrix}$$

$$\text{So, } A^{-1} = \frac{1}{\det(A)} \tilde{A} = \frac{1}{4}\begin{pmatrix} 4 & -1 & -2 \\ -4 & 2 & 4 \\ -4 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1/4 & -1/2 \\ -1 & 1/2 & 1 \\ -1 & -1/4 & 1/2 \end{pmatrix}$$ //

5. Let $T : R^3 \to R^3$ be the reflection about the plane $x - y = 0$. Find: (20 pts)

(a) The matrix A that describes the transformation T.

$$A = \frac{1}{1^2+(-1)^2+0^2}\begin{pmatrix} -1^2+(-1)^2+0^2 & -2\cdot 1(-1) & -2\cdot 1\cdot 0 \\ -2\cdot 1\cdot(-1) & 1^2-(-1)^2+0^2 & -2(-1)\cdot 0 \\ -2\cdot 1\cdot 0 & -2(-1)\cdot 0 & 1^2+(-1)^2+0^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ //

(b) The *kernel* of the transformation T.

$$A\vec{x} = \vec{0} \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \left\{ \begin{matrix} x_2 = 0 \\ x_1 = 0 \\ x_3 = 0 \end{matrix} \right\} \Rightarrow \vec{u} = \vec{0}$$

So, $\ker T = \{\vec{0}\}$ //

(c) All the *fixed points* under T.

$$A\vec{x}=\vec{x} \Rightarrow \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \Rightarrow \begin{Bmatrix}x_2=x_1\\x_1=x_2\\x_3=x_3\end{Bmatrix} \Rightarrow x_1=x_2 \longrightarrow \vec{u}=\begin{pmatrix}x_2\\x_2\\x_3\end{pmatrix}$$

So, $\mathrm{Fix}\,T=\left\{\begin{pmatrix}x_2\\x_2\\x_3\end{pmatrix}\right\}$ //

(d) The image of the plane $2x-y+z=3$ under T.

The vector eqn of the plane above is: $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\-3\\0\end{pmatrix}+t\begin{pmatrix}1\\2\\0\end{pmatrix}+s\begin{pmatrix}0\\1\\1\end{pmatrix}$

So, $A\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\left[\begin{pmatrix}0\\-3\\0\end{pmatrix}+t\begin{pmatrix}1\\2\\0\end{pmatrix}+s\begin{pmatrix}0\\1\\1\end{pmatrix}\right]=\begin{pmatrix}-3\\0\\0\end{pmatrix}+t\begin{pmatrix}2\\1\\0\end{pmatrix}+s\begin{pmatrix}1\\0\\1\end{pmatrix}$, which is given in analytic form by:

$x-2y-z=-3$ //

6. (a) Find the 3×3 matrix that rotates a vector by $45°$ around the x-axis, and then projects it on the yz-plane. (10 pts)

$$M=\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&\sqrt{2}/2&-\sqrt{2}/2\\0&\sqrt{2}/2&\sqrt{2}/2\end{pmatrix}=\begin{pmatrix}0&0&0\\0&\sqrt{2}/2&-\sqrt{2}/2\\0&\sqrt{2}/2&\sqrt{2}/2\end{pmatrix}$$ //

(b) Use the matrix above, to find the image of the vector $\vec{u}=\begin{pmatrix}1\\1\\1\end{pmatrix}$.

$$\begin{pmatrix}0&0&0\\0&\sqrt{2}/2&-\sqrt{2}/2\\0&\sqrt{2}/2&\sqrt{2}/2\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\\sqrt{2}\end{pmatrix}$$ //

7. Consider the triangle with vertices $A=(1,0,0)$, $B=(3,0,0)$, and $C=(2,0,2)$.

(a) Find the 4×4 matrix that first rotates the triangle ABC by $90°$ about the z-axis, and then translates it by $(0,-5,0)$.

$$M=\begin{pmatrix}1&0&0&0\\0&1&0&-5\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}0&-1&0&0\\1&0&0&-5\\0&0&1&0\\0&0&0&1\end{pmatrix}$$ //

(b) Use the above matrix to find the coordinates of the shifted triangle.

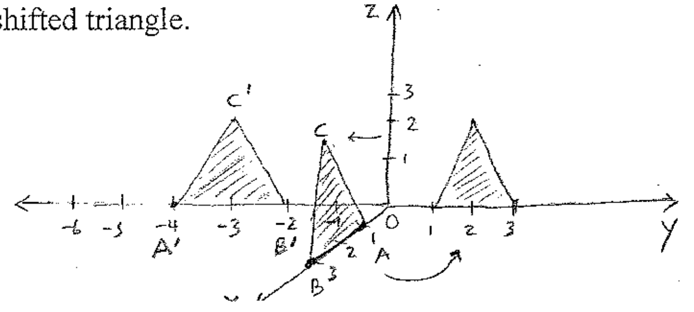
$$A'=MA=\begin{pmatrix}0&-1&0&0\\1&0&0&-5\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\-4\\0\\1\end{pmatrix}$$

$$C'=MC=\begin{pmatrix}0&-1&0&0\\1&0&0&-5\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}2\\0\\2\\1\end{pmatrix}=\begin{pmatrix}0\\-3\\2\\1\end{pmatrix}$$

$$B'=MB=\begin{pmatrix}0&-1&0&0\\1&0&0&-5\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}3\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\-2\\0\\1\end{pmatrix}$$

ie, $A'=(0,-4,0)$
$B'=(0,-2,0)$
$C'=(0,-3,2)$ //

(c) Make all relevant graphs that show the original triangle, the rotation, and the shifted triangle. (15 pts)



Some useful formulas:

(1) **Reflection** about the plane $ax+by+cz=0$:

$$A=\frac{1}{a^2+b^2+c^2}\begin{pmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{pmatrix}$$

(2) **Projection** onto the plane $ax+by+cz=0$:

$$A=\frac{1}{a^2+b^2+c^2}\begin{pmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{pmatrix}$$

(3) **Rotation** about the unit vector $\vec{u}=\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ by an angle θ :

$$A=\begin{pmatrix} a^2(1-\cos\theta)+\cos\theta & ab(1-\cos\theta)-c\sin\theta & ac(1-\cos\theta)+b\sin\theta \\ ab(1-\cos\theta)+c\sin\theta & b^2(1-\cos\theta)+\cos\theta & bc(1-\cos\theta)-a\sin\theta \\ ac(1-\cos\theta)-b\sin\theta & bc(1-\cos\theta)+a\sin\theta & c^2(1-\cos\theta)+\cos\theta \end{pmatrix}$$