

TEST 2

(Math 140-C)

Solve the following problems. Show all your work in the space under each problem.

1. (a) **True or False:** The vector $\vec{u} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$. (15 pts)

True, bec. $\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} = 1 \cdot \begin{pmatrix} -3 \\ 3 \end{pmatrix}$ multiple of \vec{u} .

- (b) Find the eigenvalues of $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$.

$$\det(A - \lambda I) = 0 \Rightarrow \det \begin{pmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{pmatrix} = 0 \Rightarrow (4-\lambda)(2-\lambda) - 3 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 5 = 0 \Rightarrow (\lambda-1)(\lambda-5) = 0 \quad \left\langle \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = 5 \end{array} \right.$$

- (c) Find the corresponding eigenvectors for each eigenvalue of A .

For $\lambda_1 = 1$: $\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases} 4x_1 + 3x_2 = x_1 \\ x_1 + 2x_2 = x_2 \end{cases} \Rightarrow \begin{cases} 3x_1 + 3x_2 = 0 \\ x_1 + x_2 = 0 \end{cases} \rightarrow \begin{cases} x_1 + x_2 = 0 \\ x_1 = -x_2 \end{cases}$

So, $\vec{u}_1 = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = 5$: $\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 5 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases} 4x_1 + 3x_2 = 5x_1 \\ x_1 + 2x_2 = 5x_2 \end{cases} \Rightarrow \begin{cases} -x_1 + 3x_2 = 0 \\ x_1 - 3x_2 = 0 \end{cases} \Rightarrow x_1 = 3x_2$

So, $\vec{u}_2 = \begin{pmatrix} 3x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

2. Given that $\vec{u} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$, then: (15 pts)

- (a) Find the cross product of \vec{u} and \vec{v} .

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 3 & -1 & 0 \\ 2 & 0 & 2 \end{vmatrix} = \vec{e}_1(-2) - \vec{e}_2(6) + \vec{e}_3(2) = \begin{pmatrix} -2 \\ -6 \\ 2 \end{pmatrix}$$

- (b) **True or False:** \vec{u} and \vec{v} are parallel.

False, bec. $\vec{u} \times \vec{v} \neq \vec{0}$

- (c) Find the area of the parallelogram generated by \vec{u} and \vec{v} .

$$\text{Area} = \|\vec{u} \times \vec{v}\| = \sqrt{(-2)^2 + (-6)^2 + 2^2} = \sqrt{44}$$

3. Given the plane $M: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix}$ and the line $l: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} -2 \\ -6 \\ -2 \end{pmatrix}$:

(a) Express the plane M in the form $ax+by+cz=d$. (10 pts)

From M , we have: $\vec{u} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix}$, and $\rho = (-1, 1, 2)$.

So, $\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 3 & 0 \\ 2 & 7 & 1 \end{vmatrix} = \vec{e}_1(3) - \vec{e}_2(1) + \vec{e}_3(1) = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

Hence, $M: 3(x+1) + (-1)(y-1) + 1(z-2) = 0 \Rightarrow 3x - y + z = -2$ //

(b) Find the point of intersection of the plane M and the line l .

From l , we have: $x = 2 - 2t$, $y = 2 - 6t$, $z = -1 - 2t$

Putting l in M : $3(2-2t) - (2-6t) + (-1-2t) = -2$

$\Rightarrow 6 - 6t - 2 + 6t - 1 - 2t = -2 \rightarrow t = 5/2$

So, for $t = 5/2$ in l , we get: $Q = (-3, -13, -6)$ //

4. Given $A = \begin{pmatrix} 2 & 0 & 7 \\ 0 & -1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, find: (15 pts)

(a) The $\det(A)$.

$\det(A) = 2(-5) - 0(-4) + 7(2) = 4$ //

(b) Find the cofactor matrix of A .

$C_A = \begin{pmatrix} \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 2 & 2 \end{vmatrix} \\ -\begin{vmatrix} 0 & 7 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 7 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 2 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & 7 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 7 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -5 & 4 & 2 \\ 14 & -12 & -4 \\ 7 & -4 & -2 \end{pmatrix}$ //

(c) Find A^{-1} using the adjoint of A .

$\tilde{A} = C_A^t = \begin{pmatrix} -5 & 14 & 7 \\ 4 & -12 & -4 \\ 2 & -4 & -2 \end{pmatrix}$

So, $A^{-1} = \frac{1}{\det(A)} \tilde{A} = \frac{1}{4} \begin{pmatrix} -5 & 14 & 7 \\ 4 & -12 & -4 \\ 2 & -4 & -2 \end{pmatrix} = \begin{pmatrix} -5/4 & 14/4 & 7/4 \\ 1 & -3 & -1 \\ 1/2 & -1 & -1/2 \end{pmatrix}$ //

5. Let $T: R^3 \rightarrow R^3$ be the reflection about the plane $x+z=0$. Find: (20 pts)

(a) The matrix A that describes the transformation T .

$A = \frac{1}{1^2+0^2+1^2} \begin{pmatrix} -1^2+0^2+1^2 & -2 \cdot 1 \cdot 0 & -2 \cdot 1 \cdot 1 \\ -2 \cdot 1 \cdot 0 & 1^2-0^2+1^2 & -2 \cdot 0 \cdot 1 \\ -2 \cdot 1 \cdot 1 & -2 \cdot 0 \cdot 1 & 1^2+0^2-1^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ //

(b) The kernel of the transformation T .

$A\vec{x} = \vec{0} \Rightarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -x_3 = 0 \\ x_2 = 0 \\ -x_1 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases} \rightarrow \vec{u} = \vec{0}$

So, $\text{Ker } T = \{\vec{0}\}$ //

(c) All the fixed points under T .

$$A\vec{x} = \vec{x} \Rightarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{cases} -x_3 = x_1 \\ x_2 = x_2 \\ -x_1 = x_3 \end{cases} \Rightarrow x_1 = -x_3 \rightarrow \vec{u} = \begin{pmatrix} -x_3 \\ x_2 \\ x_3 \end{pmatrix}$$

So, $\text{Fix } T = \left\{ \begin{pmatrix} -x_3 \\ x_2 \\ x_3 \end{pmatrix} \right\}$.

(d) The image of the plane $x - 2y + 3z = 1$ under T .

The vector eqn of the plane above is: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

So, $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \left[\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, which is given in analytic form by:

6. (a) Find the 3×3 matrix that rotates a vector by 30° around the z -axis, and then projects it on the xy -plane. (10 pts)

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) Use the matrix above, to find the image of the vector $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$$\begin{pmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}-1}{2} \\ \frac{1+\sqrt{3}}{2} \\ 0 \end{pmatrix}$$

7. Consider the triangle with vertices $A = (0,1,0)$, $B = (0,3,0)$, and $C = (0,2,2)$.

(a) Find the 4×4 matrix that first rotates the triangle ABC by 90° about the z -axis, and then translates it by $(-4,0,0)$.

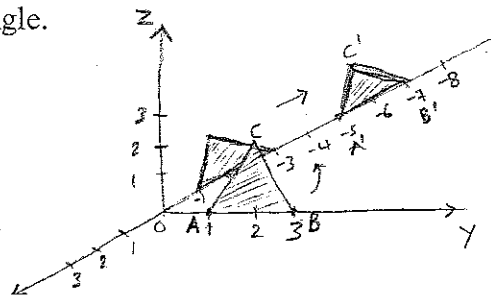
$$M = \begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) Use the above matrix to find the coordinates of the shifted triangle.

$$A' = MA = \begin{pmatrix} 0 & -1 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad C' = MC = \begin{pmatrix} 0 & -1 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

$$B' = MB = \begin{pmatrix} 0 & -1 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{ie, } \begin{matrix} A' = (-5, 0, 0) \\ B' = (-7, 0, 0) \\ C' = (-6, 0, 2) \end{matrix}$$

(c) Make all relevant graphs that show the original triangle, the rotation, and the shifted triangle. (15 pts)



Some useful formulas:

(1) **Reflection** about the plane $ax + by + cz = 0$:

$$A = \frac{1}{a^2 + b^2 + c^2} \begin{pmatrix} -a^2 + b^2 + c^2 & -2ab & -2ac \\ -2ab & a^2 - b^2 + c^2 & -2bc \\ -2ac & -2bc & a^2 + b^2 - c^2 \end{pmatrix}$$

(2) **Projection** onto the plane $ax + by + cz = 0$:

$$A = \frac{1}{a^2 + b^2 + c^2} \begin{pmatrix} b^2 + c^2 & -ab & -ac \\ -ab & a^2 + c^2 & -bc \\ -ac & -bc & a^2 + b^2 \end{pmatrix}$$

(3) **Rotation** about the unit vector $\vec{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ by an angle θ :

$$A = \begin{pmatrix} a^2(1 - \cos\theta) + \cos\theta & ab(1 - \cos\theta) - c \sin\theta & ac(1 - \cos\theta) + b \sin\theta \\ ab(1 - \cos\theta) + c \sin\theta & b^2(1 - \cos\theta) + \cos\theta & bc(1 - \cos\theta) - a \sin\theta \\ ac(1 - \cos\theta) - b \sin\theta & bc(1 - \cos\theta) + a \sin\theta & c^2(1 - \cos\theta) + \cos\theta \end{pmatrix}$$