Key

Midterm Exam

Math 258
(Spring 2005)

Solve the following problems. Show all your work in the space under each problem.

1. Determine the truth value of the following statements, if the universe of discourse consists of all integers. Explain. (20 pts)

   (a) $\forall n(2n > n)$ F

   $n = -1$

   (b) $\exists n(n^3 = -1)$ T

   $n = -1$

   (c) $\exists n \forall m(nm = m)$ T

   $n = 1$

   (d) $\exists n \exists m[(n + m = 4) \wedge (n - m = 1)]$ F

   $n + m = 4$
   $n - m = 1$ +
   $2n = 5 \Rightarrow n = 2/5 \notin \mathbb{Z}$

2. Prove that the sum of an irrational number and a rational number is irrational. (*Hint*: Use a proof by contradiction) (10 pts)

   Let r be rat. and i be irrat. Then $r = p/q$
   Supp. $r + i$ is rat, ie $r + i = p'/q'$
   $\Rightarrow p/q + i = p'/q'$
   $\Rightarrow i = p'/q' - p/q \Rightarrow i = \frac{p'q - q'p}{q'q} \rightarrow$ rational ↯

3. (a) Find two sets A and B such that $A \in B$ and $A \subset B$. (10 pts)

   $A = \{\emptyset\}$, $B = \{\emptyset, \{\emptyset\}\}$

   (b) Find the power set of the set $\{\emptyset, \{\emptyset, \{\emptyset\}\}\}$.

   $P = \{\emptyset, \{\emptyset\}, \{\{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\}$

4. The **successor** of a set A is the set $A \cup \{A\}$. Find the successor of the set: (10 pts)

   $$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

   How many elements does the successor of a set with n elements have?

   $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$

   $n + 1$

5. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

(10 pts)

"$\Rightarrow$" Let $x \in \overline{B} \Rightarrow x \notin B$

$\Rightarrow x \notin A$, bec. if $x \in A \overset{A \subset B}{\Rightarrow} x \in B$ ↯

$\Rightarrow x \in \overline{A}$

ie, $\overline{B} \subset \overline{A}$

"$\Leftarrow$" Let $x \in A \Rightarrow x \notin \overline{A}$

$\Rightarrow x \notin \overline{B}$, bec. if $x \in \overline{B} \overset{\overline{B} \subset \overline{A}}{\Rightarrow} x \in \overline{A}$ ↯ , ie $A \subset B$

$\Rightarrow x \in B$

6. Show that the following function is a bijection: $f: \mathbf{R} \to \mathbf{R}$

$f(x) = 2x - 3$

(10 pts)

1-1 : Let $f(x_1) = f(x_2) \Rightarrow 2x_1 - 3 = 2x_2 - 3$

$\Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$ //

onto : $\forall y \ \exists x = \frac{y+3}{2}$ st $f(x) = y$

(Indeed, $f(x) = f\left(\frac{y+3}{2}\right) = \cancel{2}\,\frac{y+3}{\cancel{2}} - 3 = y + \cancel{3} - \cancel{3} = y$)

7. (a) If $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ and $AB = 0$, show that $A = 0$.

(30 pts)

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$AB = 0 \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a & 2a-b \\ c & 2c-d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \longrightarrow \left.\begin{matrix} a = 0 \\ 2a - b = 0 \\ c = 0 \\ 2c - d = 0 \end{matrix}\right\} \Rightarrow a = b = c = d = 0$, ie $A = 0$ //

(b) Construct an example where $AB = 0$, but both A and B are <u>not</u> zero.

Take $\left.\begin{matrix} A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq 0 \\ B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq 0 \end{matrix}\right\} \longrightarrow AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ //

(c) Given $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, show that $A^3 = I$. Use that to find A^{-1}.

(*Note*: Do <u>not</u> use the formula for A^{-1} to find A^{-1})

$A^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$

$A^3 = A^2 A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Since $A^3 = I \Rightarrow A^2 A = I \Rightarrow A^{-1} = A^2$.