TEST 1

(Math 258, A)

1. Let $P(x)$ be the statement "$x = x^2$." If the universe of discourse consists of the integers, what is the truth value of the following? Explain. (15 pts)

(a) $P(-1)$

$-1 = (-1)^2$

$-1 = 1$

False

(b) $\exists x P(x)$

$1 = 1^2$

$1 = 1$

True

(c) $\forall x P(x)$

$3 = 3^2$

$3 = 9$

False

2. Determine the truth value of the following statements, if the universe of discourse consists of all real numbers. Explain. (15 pts)

(a) $\forall x \exists y (x = y^2)$

$-2 = (\sqrt{-2})^2$

But $y = (\sqrt{-2})$ isn't a real number

False

(b) $\forall x \exists y (xy = 0)$

$x \cdot 0 = x$ for all x

True

(c) $\exists x \exists y [(x + 2y = 2) \wedge (2x + 4y = 5)]$

$x + 2y = 2$ implies that $x = 2 - 2y$

Putting that into the second equation yields

$2(2 - 2y) + 4y = 5$

$4 - 4y + 4y = 5$

$4 = 5$

False

3. Determine whether the following arguments are valid, giving counterexamples where necessary: (10 pts)

(a) If x is a real number such that $x > 5$, then $x^2 > 25$.
Suppose that $x \leq 5$, then $x^2 \leq 25$.

Invalid
$x = -6$ implies $x^2 = 36 \geq 25$

(b) If x^2 is irrational, then x is irrational.
Suppose x is irrational, then x^2 is irrational.

Invalid
Suppose x is $\sqrt{2}$, which is irrational. Then $x^2 = 2$, which is not irrational.

4. Use a truth table to show that the following argument is valid: (15 pts)

$$p \rightarrow q$$
$$p \rightarrow r$$
$$\neg(p \wedge q)$$

$$\neg p$$

p	q	r	$p \rightarrow q$	$p \rightarrow r$	$\neg(p \wedge q)$	$(p \rightarrow q) \wedge (p \rightarrow r) \wedge (\neg(p \wedge q))$	$\neg p$
T	T	T	T	T	F	F	F
T	T	F	T	F	F	F	F
T	F	T	F	T	T	F	F
T	F	F	F	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since for all cases that the hypotheses are True (that is rows 5-8), we have that the conclusion is True, we have that the argument is **valid**.

5. Prove, or disprove, that the product of two irrational numbers is irrational. (10 pts)
(*Note*: If you are disproving, provide a counterexample)

Disprove
$\sqrt{2}$ is irrational. But $\sqrt{2} \cdot \sqrt{2} = 2$, which is rational. Disproved by counterexample.

6. Prove that if $n^3 + 5$ is odd, where n is an integer, then n is even. (10 pts)
(*Hint*: Use proof by contradiction)

Suppose n is odd. Then $n = 2k + 1$. Then
$n^3 + 5 = (2k + 1)^3 + 5$
$= 8k^3 + 12k^2 + 6k + 6$
$= 2\,(4k^3 + 6k^2 + 3k + 3)$ which is even.

Done.

7. Prove that if a, b and c are real numbers, then $\min(a, \min(b,c)) = \min(c, \min(a,b))$.
(*Hint*: Use proof by cases) (10 pts)

if $a < b < c$
$\min(a, \min(b, c)) = \min(a, b) = a$
$\min(c, \min(a, b)) = \min(c, a) = a$

if $a < c < b$
$\min(a, \min(b, c)) = \min(a, c) = a$
$\min(c, \min(a, b)) = \min(c, a) = a$

if $b < a < c$
$\min(a, \min(b, c)) = \min(a, b) = b$
$\min(c, \min(a, b)) = \min(c, b) = b$

if $b < c < a$
$\min(a, \min(b, c)) = \min(a, b) = b$
$\min(c, \min(a, b)) = \min(c, b) = b$

if $c < b < a$
$\min(a, \min(b, c)) = \min(a, c) = c$
$\min(c, \min(a, b)) = \min(c, b) = c$

if $c < a < b$
$\min(a, \min(b, c)) = \min(a, c) = c$
$\min(c, \min(a, b)) = \min(c, a) = c$

8. Show that, for integers, the following statements are equivalent: (15 pts)
(*Hint*: consider proof by contradiction for some of your steps)

(i) $3n + 2$ is even
(ii) $n + 5$ is odd

(iii) n^2 even.

(i) -> (ii) $3n + 2$ is even implies $n + 5$ is odd

Proof by contradiction.
Suppose $n + 5$ is even. Then
$n + 5 = 2k$
$n = 2k - 5$
$n = 2k - 6 + 1$
$n = 2(k - 3) + 1$
$n = 2(p) + 1$ which is odd.

Then
$3n + 2 = 3(2p + 1) + 2$
$= 6p + 3 + 2$
$= 6p + 5$
$= 6p + 4 + 1$
$= 2(3p + 2) + 1$
$= 2q + 1$ which is odd

Done.

(ii) -> (iii) $n + 5$ is odd implies n^2 even

$n + 5$ is odd implies that
$n + 5 = 2k + 1$
$n = 2k - 4$
$n = 2p$

Then
$n^2 = (2p)^2$
$n^2 = 4p$
$n^2 = 2(2p)$
$n^2 = 2q$ which is even

Done.

(iii) -> (i) n^2 even implies $3n + 2$ is even

n^2 even implies that n is even (proven in class). So, n = $2k$.
Then
$3n + 2 = 3(2k) + 2$
$3n + 2 = 6k + 2$
$3n + 2 = 2(3k + 1)$ which is even

Done.