TEST 1 - Key

(Math 258 , B)

1. Let $P(x)$ be the statement "$x+1=2$." If the universe of discourse consists of the integers, what is the truth value of the following? Explain. (15 pts)

   (a) $P(0)$ (b) $\exists x P(x)$ (c) $\forall x P(x)$

(a) **False**, $\text{P}(0)=1\neq 2$
(b) **True**, there exists an x, namely, 1, st 1 + 1 = 2
(c) **False**, not every x satisfies that statement (ie let 'x' is anything but 1)

2. Determine the truth value of the following statements, if the universe of discourse consists of all real numbers. Explain. (15 pts)

   (a) $\forall x \exists y (x = y^2)$ (b) $\forall x \exists y (xy = x)$ (c) $\exists x \exists y [(x+3y=1) \wedge (2x+6y=3)]$

(a) **False**, let $x=-1$ then $-1=y^2$, hence $y=\sqrt{-1}$, which is not Real
(b) **True**, letting y be the multiplicative identity 1, x*y = x (under R by definition)
(c) **False**,

$x+3y-1=2x+6y-3$
(solving for x)
$x=-3y+2$
(substituting for 'x')
$(-3y+2)+3y=1$
$2=1$, false statement under R, hence False

3. Determine whether the following arguments are valid, giving counterexamples where necessary: (10 pts)

   (a) If x is a real number such that x > 3, then x^2 > 9.
   Suppose that $x \leq$ 3, then $x^2 \leq$ 9.

Invalid
$p = x > 3$
$q = x^2 > 9$
We are supposing, $\neg p \rightarrow \neg q$, Hence Invalid

   (b) If x^2 is irrational, then x is irrational.

Suppose x is irrational, then x^2 is irrational.

Invalid
$p = x^2$ is irrational
$q = x$ is irrational
We are supposing, $q \to p$, Hence Invalid

4. Use a truth table to show that the following argument is valid: (15 pts)

$$\neg r$$
$$p \to r$$
$$q \to r$$

$$\neg(p \wedge q)$$

p	q	r	$\neg r$	$p \to r$	$q \to r$	$(\neg r) \wedge (p \to r)$	$(\neg r) \wedge (p \to r) \wedge (q \to r)$	$p \wedge q$	$\neg(p \wedge q)$	$(\neg r) \wedge (p \to r) \wedge (q \to r) \to \neg(p \wedge q)$
T	T	T	F	T	T	F	F	T	F	**T**
T	T	F	F	F	F	F	F	T	F	**T**
T	F	T	T	T	T	T	T	F	T	**T**
T	F	F	T	F	T	F	F	F	T	**T**
F	T	T	F	T	T	F	F	F	T	**T**
F	T	F	F	T	F	F	F	F	T	**T**
F	F	T	T	T	T	T	T	F	T	**T**
F	F	F	T	T	T	T	T	F	T	**T**

Last column is all true (a **tautology**) showing that the argument is valid

5. Prove, or disprove, that the product of two irrational numbers is irrational. (10 pts)
(*Note*: If you are disproving, provide a counterexample)
Use a Direct Proof to **Disprove**
Let i, j∈ Irrationals, s.t.
i = $\sqrt{2}$
j = $\sqrt{2}$

Then, the product of i and j = $\sqrt{2} * \sqrt{2}$ = 2, which is rational, **disproving** the statement

6. Prove that if $n^3 + 5$ is odd, where n is an integer, then n is even. (10 pts)
(*Hint*: Use proof by contradiction)

Assume that n is odd, then
$n = (2k+1)$, so
$n^3 + 5 = (2k+1)^3 + 5$
$= 8x^3 + 12x^2 + 6x + 1 + 5$
$= 2(4x^3 + 6x^2 + 3x + 3)$
$=$ 2k, where k $= 4x^3 + 6x^2 + 3x + 3)$

Hence, $n^3 + 5$ is even, **contradiction**, proving $p \to q$

7. Prove that if a, b and c are real numbers, then $\min(a, \min(b,c)) = \min(c, \min(a,b))$.
(*Hint*: Use proof by cases) (10 pts)

case 1: $a \le b, c$
Let n = $\min(b,c)$, $a \le n$
$\min(a, \min(b,c)) = \min(a,n) = a$
$\min(c, \min(a,b)) = \min(c,a) = a$

case 2: $b \le c, a$
$\min(a, \min(b,c)) = \min(a,b) = b$
$\min(c, \min(a,b)) = \min(c,b) = b$

case 3: $c \le a, b$
Let n = $\min(a,b)$, $c \le n$
$\min(a, \min(b,c)) = \min(a,c) = c$
$\min(c, \min(a,b)) = \min(a,n) = c$

All three cases are equal, hence if a, b and c are real numbers, then
$\min(a, \min(b,c)) = \min(c, \min(a,b))$

8. Show that, for integers, the following statements are equivalent: (15 pts)
(*Hint*: consider proof by contradiction for some of your steps)

(i) $3n + 2$ is even
(ii) $n + 5$ is odd
(iii) n^2 even.

(need to show: $(i) \to (ii) \to (iii) \to (i)$)

$(i) \to (ii)$
Suppose $n + 5$ is even, then
$\Rightarrow n + 5 = 2$k

$\Rightarrow$ $n = 2k - 5$

$\Rightarrow$ $\quad = 2k - 6 + 1$

$\Rightarrow$ $\quad = 2m + 1$, *where* $m = (k - 3)$

ie, n is odd, so

$\Rightarrow$ $3(2m + 1) + 2 = 6m + 3 + 2$

$\Rightarrow$ $\qquad = 6m + 4 + 1$

$\Rightarrow$ $\qquad = 2z + 1$, where $z = (3m + 2)$

$\Rightarrow$ hence, $3n + 2$ is odd, contradiction

$(ii) \rightarrow (iii)$

$n + 5$ is odd, so

$\Rightarrow$ $n + 5 = 2k + 1$

$\Rightarrow$ $n = 2k - 4$

$= 2m$, where $m = (k - 2)$

Hence, n is even, so n^2 is even

$(iii) \rightarrow (i)$

Suppose $3n + 2$ is odd, so

$\Rightarrow$ $3n + 2 = 2k + 1$

$\Rightarrow$ $3n + 3 = 2k + 2$

$\Rightarrow$ $3(n + 1) = 2m$, where $m = (k + 1)$

hence, $3(n + 1)$ is even

$\Rightarrow$ $(n + 1)$ even

$\Rightarrow$ n odd

$\Rightarrow$ n^2 odd

<u>or</u>

n^2 even implies n is even

ie, $n = 2k$

$\Rightarrow$ $3n + 2 = 3(2k) + 2$

$\Rightarrow$ $\qquad = 6k + 2$

$\Rightarrow$ $\qquad = 2m$, where $m = 3k + 1$

Hence, $3n + 2$ is even