Homework 1

(Math 140)

1. **What is the dimension of C, the vector space of the complex numbers? Why?**

We claim that 1 and i are a basis for **C.**

(a) First we will show that they are linearly independent:
Let $a,b \in R$, $a1 + bi = 0$. The only way this is true is when a = b = 0.
So they are linearly independent.

(b) Now we show that they span **C.**
Any $a + bi$ in **C**, can be written as $a + bi = a1 + bi$.
So, they span **C**.

Hence, the dimension of **C** is **two**.

2. **Show that the polynomials** $p_1(x) = 1 + x,\ p_2(x) = 1 - x,\ p_3(x) = x + x^2$ **form a basis for the vector space** $P_2[x]$**, the space of second-degree polynomials.**

One can easily show, as in problem 1, that p_1, p_2, p_3 are linearly independent and span $P_2[x]$. Or, one can do the following:
Write the vectors p_1, p_2, p_3 as columns of a matrix. If the determinant is not zero, the vectors are linearly independent. In other words:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \text{A}$$

Det(A) = $1(-1-0) - 1(1-0) - 0 = -2 \neq 0$

Since p_1, p_2, p_3 are 3 linearly independent vectors in a 3-dimensional space, they are a basis.

3. **Given** $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$**, find a formula for** A^n**, where n is a positive integer.**

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$
$$A^3 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$$
$$A^4 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}$$

.
.
.
etc

So, $A^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$

4. **Given show that** $A^3 = I$**. Use that to find** A^{-1}**.**

$$A^3 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^2 A = I \Rightarrow A^{-1} = A^2$$

5. **Given A and B are n x n matrices, show that (A + B)(A – B) =** $A^2 - B^2$ **if and only if AB = BA.**

"$\Rightarrow$" $(A+B)(A-B) = A^2 - AB + BA - B^2 = A^2 - B^2$ (since AB = BA)

"$\Leftarrow$" $(A+B)(A-B) = A^2 - AB + BA - B^2 = A^2 - B^2 \Rightarrow AB = BA$

6. **Given the map** $A: R^2 \to R^2$**, defined by** $A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 + u_2 \\ -u_2 \end{pmatrix}$ **do the following:**

   a. ***Show that A is a linear map***

   Let $x, y \in R^2$, $c \in R$. Where $x = (a_1, a_2), y = (b_1, b_2)$

   $cx + y = (ca_1 + b_1, ca_2 + b_2)$

   $A(cx + y) = A(ca_1 + b_1, ca_2 + b_2)$

   $= (ca_1 + b_1 + ca_2 + b_2, -ca_2 - b_2)$

   $cA(x) + A(y) = c(a_1 + a_2, -a_2) + (b_1 + b_2, -b_2)$

   $= (ca_1 + b_1 + ca_2 + b_2, -ca_2 - b_2)$

   b. ***Find K_A, the kernel of A***

   $$A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} u_1 + u_2 = 0 \\ -u_2 = 0 \end{matrix} \Rightarrow u_1 = u_2 = 0 \Rightarrow K_A = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

   c. ***Find the matrix that expresses the linear map A***

   $$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

7. **If the area of a parallelogram, generated by** $\vec{u}$ **and** $\vec{v}$**, is equal to 2 square units, then what is the area of the new parallelogram formed after we apply matrix** $A = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$ **on vectors** $\vec{u}$ **and** $\vec{v}$**?**

   $\det(A) = 4 - 3 = 1$

   Since the determinant is 1, the area is unchanged. So, the area is 2.