Homework 1 - Key

MAT 258

1.Let $f(x) = ax + 1$ and $g(x) = 2x + a + 2$, where a is a real number. Find for which a's we have $f \circ g = g \circ f$.

$$f \circ g = f(g(x)) = f(2x + a + 2) = a(2x + a + 2) + 1 = 2ax + a^2 + 2a + 1$$
$$g \circ f = g(f(x)) = g(ax + 1) = 2(ax + 1) + a + 2 = 2ax + 2 + a + 2$$

Want to find a's such that : $2ax + a^2 + 2a + 1 = 2ax + 2 + a + 2$

The equation above gives: $a^2 + a - 3 = 0$, which has solution $a = \dfrac{-1 \pm \sqrt{13}}{2}$.

2.Let f: A → B and $S \subseteq$ A, $T \subseteq$ A. Show that:

(a) $f(S \cup T) = f(S) \cup f(T)$

"$\Rightarrow$" Let $y \in f(S \cup T)$
$\Rightarrow \exists\, x \in S \cup T$ st $f(x) = y$
$\Rightarrow \exists\, x \in S$ or $x \in T$ st $f(x) = y$
$\Rightarrow \exists\, x \in S$ st $f(x) = y$ or $\exists\, x \in T$ st $f(x) = y$
$\Rightarrow y \in f(S)$ or $y \in f(T)$
$\Rightarrow y \in f(S) \cup f(T)$

"$\Leftarrow$" Let $y \in f(S) \cup f(T)$
$\Rightarrow y \in f(S)$ or $y \in f(T)$
$\Rightarrow \exists\, x \in S, f(x) = y$ or $\exists\, x' \in T, f(x') = y$
$\Rightarrow \exists\, x \in S$ or T st $f(x) = y$
$\Rightarrow \exists\, x \in S \cup T$ st $f(x) = y$
$\Rightarrow \exists\, y \in f(S \cup T)$

(b) $f(S \cap T) \subseteq f(S) \cap f(T)$

Let $y \in f(S \cap T)$
$\Rightarrow \exists\, x \in S \cap T$ st $f(x) = y$
$\Rightarrow \exists\, x \in S$ and $x \in T$ st $f(x) = y$
$\Rightarrow \exists\, x \in S$ st $f(x) = y$ and $\exists\, x \in T$ st $f(x) = y$
$\Rightarrow y \in f(S)$ and $y \in f(T)$
$\Rightarrow y \in f(S) \cap f(T)$

(c)Give an example of a function *f* and sets *S* and *T* such that $f(S \cap T) \neq f(S) \cap f(T)$

Let $S = \{1\}$, $T = \{2\}$, and $B = \{3\}$
Let $f : A \rightarrow B$, defined by:
$f(1) = 3$ and $f(2) = 3$
Then, $S \cap T = \varnothing$
So, $f(S \cap T) = f(\varnothing) = \varnothing$ and $f(S) \cap f(T) = B = \{3\}$

The above shows: $f(S \cup T) \neq f(S) \cap f(T)$

3.Let $f : A \rightarrow B$ and $S \subseteq B$. The **inverse image** of *S* is defined by:
$$f^{-1}(S) = \{ a \in A \mid f(a) \in S \}$$

In other words, is the subset of *A* containing the pre-images of all elements of S. Let *f* be a real function defined by $f(x) = x + 1$. Find:

(a)$f^{-1}(\{0\}) = \{ a \mid a = -1 \}$
(b)$f^{-1}(\{x \mid 0 < x < 1\}) = \{ a \mid -1 < a < 0 \}$

4.Given $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, find a formula for A^n, where *n* is a positive integer.

$$A^2 = AA = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = A\,A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$$

Hence, $A^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$

5.Given $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, show that $A^3 = I$. Use that to find A^{-1}

$$A^3 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence, $A^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = A^2$

6.Given that A and B and *n* x *n* matrices, show that $(A + B)(A - B) = A^2 - B^2$ if and only if AB = BA

"$\Leftarrow$" $(A + B)(A - B) = A^2 - AB + BA - B^2$

$= A^2 - B^2$ (since AB = BA)

"$\Rightarrow$" $(A + B)(A - B) = A^2 - B^2$

$= A^2 - AB + BA - B^2 = A^2 - B^2$

$= -AB + BA = 0$

$= AB = BA$