Test 2 (Answer Key)

(Math 140)

1. Find the angle between the vectors $\vec{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Are these two vectors parallel? Explain.

   We know that $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$

   We have $|\vec{u}| = \sqrt{2}$, $|\vec{v}| = \sqrt{2}$ and $\vec{u} \cdot \vec{v} = -2$

   So, $\cos\theta = \dfrac{-2}{\sqrt{2}\sqrt{2}} = -1$. This implies that $\theta = 180°$.

   Hence, the vectors are parallel.

2. Show that the vectors $\vec{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ are linearly independent.

   For $\vec{u}$ and $\vec{v}$ to be linearly independent, $a\vec{u} + b\vec{v} = 0$ iff a = b = 0.

   $$a\begin{pmatrix} -1 \\ 2 \end{pmatrix} + b\begin{pmatrix} -3 \\ 1 \end{pmatrix} = 0 \Rightarrow \begin{cases} -a - 3b = 0 \\ 2a + b = 0 \end{cases} \Rightarrow \begin{cases} a = -3b \\ 2a + b = 0 \end{cases}$$

   Show b is 0
   $2(-3b) + b = 0 \Rightarrow -5b = 0 \Rightarrow b = 0$

   Show a is 0
   $a = -3b \Rightarrow a = 0$

   So $\vec{u}$ and $\vec{v}$ are linearly independent.

3. Show that $M_{2x2}(R)$ is spanned by the following vectors:

   $$\vec{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \vec{E}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \vec{E}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \vec{E}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Any A$=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a linear combination of $\vec{E}_1, \vec{E}_2, \vec{E}_3, \vec{E}_4$.

Indeed, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = aE_1 + bE_2 + cE_3 + dE_4$. Hence, $\vec{E}_1, \vec{E}_2, \vec{E}_3, \vec{E}_4$ span $M_{2x2}(R)$.

4. Given $A = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 3 & -1 \end{pmatrix}$, find AB + B^2 – 6I.

$$\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 3 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 3 & -1 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 6 & -3 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ -3 & 4 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 3 & -5 \end{pmatrix}$$

5. Given $A = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$, find A^{-1}.

$$A^{-1} = \frac{1}{4-2}\begin{pmatrix} 4 & -2 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1/2 & 1/2 \end{pmatrix}$$

6. Show the following map is a linear map:

$$T : R^2 \to R^2$$
$$T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}$$

(a) $T\left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right] = T\begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ -u_2 - v_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix} = T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + T\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

(b) $T\left[\lambda\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right] = T\begin{pmatrix} \lambda u_1 \\ \lambda u_2 \end{pmatrix} = \begin{pmatrix} \lambda u_1 \\ -\lambda u_2 \end{pmatrix} = \lambda\begin{pmatrix} u_1 \\ -u_2 \end{pmatrix} = \lambda T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

Hence, T is linear.

7. Identify the type of the linear map the following represent:

(a) $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (b) $B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (c) $C = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ (d) $D = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}$

(a) Scale
(b) Reflect
(c) Rotation
(d) Projection

8. Find the matrix that changes the basis $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ into $\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2\vec{u}_1 + (-1)\vec{u}_2$$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1\vec{u}_1 + 2\vec{u}_2$$

So, the required matrix is $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$.