

HOMEWORK 2

(Math 140)

1. Show that if A and B are rotations of \mathfrak{R}^2 , then $AB = BA$. Is the same true for rotations in \mathfrak{R}^3 ? (20 pts)

$$\begin{aligned} AB &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta \cos \alpha - \sin \theta \sin \alpha & -\cos \theta \sin \alpha - \sin \theta \cos \alpha \\ \sin \theta \cos \alpha + \cos \theta \sin \alpha & -\sin \theta \sin \alpha + \cos \theta \cos \alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos \alpha \cos \theta - \sin \alpha \sin \theta & -\sin \alpha \cos \theta - \sin \theta \cos \alpha \\ \cos \alpha \sin \theta + \sin \alpha \cos \theta & -\sin \alpha \sin \theta + \cos \alpha \cos \theta \end{pmatrix} \\ &= BA \end{aligned}$$

So, $AB = BA$ in \mathfrak{R}^2 .

But, this is not the case for \mathfrak{R}^3 . For example, consider the rotations:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then, $A \neq B$.

2. Given $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -3 & 2 \\ 0 & 1 & -1 \end{pmatrix}$, find A^{-1} . (20 pts)

The cofactor matrix is: $C_A = \begin{pmatrix} 1 & 2 & 2 \\ -1 & -1 & -1 \\ -2 & -2 & -1 \end{pmatrix}$

The adjoint matrix is: $\tilde{A} = (C_A)^t = \begin{pmatrix} 1 & -1 & -2 \\ 2 & -1 & -2 \\ 2 & -1 & -1 \end{pmatrix}$

Also, $\det(A) = 1(3-2) + 1(-2-0) + 0(2-0) = 1-2 = -1$

$$A^{-1} = \frac{1}{\det(A)} \tilde{A} = \frac{1}{-1} \begin{pmatrix} 1 & -1 & -2 \\ 2 & -1 & -2 \\ 2 & -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 2 \\ -2 & 1 & 2 \\ -2 & 1 & 1 \end{pmatrix}$$

3. Let A be a 3×3 matrix. Show that $|cA| = c^3|A|$. (20 pts)

Let $A = \begin{pmatrix} a & b & d \\ e & f & g \\ h & i & j \end{pmatrix}$. Then, $cA = \begin{pmatrix} ca & cb & cd \\ ce & cf & cg \\ ch & ci & cj \end{pmatrix}$

$$\begin{aligned} |cA| &= ca(cfj - cig) - cb(cej - chcg) + cd(ceci - chcf) \\ &= c^3(a(fj - ig) - b(ej - hg) + d(ei - hf)) \end{aligned}$$

$$c^3|A| = c^3(a(fj - ig) - b(ej - hg) + d(ei - hf))$$

So, $|cA| = c^3|A|$.

4. Rotation of a figure about a point P in \mathfrak{R}^2 is accomplished by first translating the figure by $-P$, rotating about the origin, and then translating back to P . Construct a 3×3 matrix A that rotates points by 45° about the point $(1,2)$, using homogeneous coordinates. (20 pts)

Let T be the matrix that translates the point $(1,2)$ to the origin. Then:

$$T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Let R be the matrix that rotates a point by 45° . Then:

$$R = \begin{pmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let S be the matrix that translates the origin to $(1,2)$. Then:

$$S = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Then, the matrix that translates the point $(1,2)$ to the origin, rotates it by 45° , and then translates it back to the point $(1,2)$ is:

$$\begin{aligned} M &= S \cdot R \cdot T \\ &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & \frac{\sqrt{2}+2}{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & \frac{-3\sqrt{2}+4}{2} \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

5. Given the basis $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, examine whether it is an orthonormal basis. If your answer is **no**, then use the *Gram-Schmidt Orthogonalization Process* to construct an orthonormal basis out of the one above. (20 pts)

This is not an orthonormal basis, because $\langle A, B \rangle = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 + 2 \cdot (-1) = -2 \neq 0$.

Let, $A' = A$.

$$\text{Then, } B' = B - \frac{\langle B, A' \rangle}{\|A'\|^2} A' = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} - \frac{-2}{5} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & 1 \\ 0 & \frac{-1}{5} \end{pmatrix}$$

$$\begin{aligned} C' &= C - \frac{\langle C, B' \rangle}{\|B'\|^2} B' - \frac{\langle C, A' \rangle}{\|A'\|^2} A' = \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix} - \frac{-1}{\frac{5}{6}} \begin{pmatrix} \frac{2}{5} & 1 \\ 0 & \frac{-1}{5} \end{pmatrix} - \frac{2}{5} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix} - \begin{pmatrix} \frac{-2}{30} & \frac{-1}{6} \\ 0 & \frac{1}{30} \end{pmatrix} - \begin{pmatrix} \frac{2}{5} & 0 \\ 0 & \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{-1}{3} & \frac{1}{6} \\ -2 & \frac{1}{6} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} D' &= D - \frac{\langle D, C' \rangle}{\|C'\|^2} C' - \frac{\langle D, B' \rangle}{\|B'\|^2} B' - \frac{\langle D, A' \rangle}{\|A'\|^2} A' \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} - \frac{2}{25} \begin{pmatrix} \frac{-1}{3} & \frac{1}{6} \\ -2 & \frac{1}{6} \end{pmatrix} - \frac{-1}{3} \begin{pmatrix} \frac{2}{5} & 1 \\ 0 & \frac{-1}{5} \end{pmatrix} - \frac{4}{5} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} \frac{-2}{75} & \frac{1}{75} \\ \frac{-4}{25} & \frac{1}{75} \end{pmatrix} + \begin{pmatrix} \frac{2}{15} & \frac{1}{3} \\ 0 & \frac{-1}{15} \end{pmatrix} - \begin{pmatrix} \frac{4}{5} & 0 \\ 0 & \frac{8}{5} \end{pmatrix} = \begin{pmatrix} \frac{-16}{25} & \frac{8}{25} \\ \frac{4}{25} & \frac{8}{25} \end{pmatrix} \end{aligned}$$

Then A', B', C', D' is an orthogonal basis .

Furthermore,

$$A'' = \frac{A'}{\|A'\|} = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix}, \quad B'' = \frac{B'}{\|B'\|} = \begin{pmatrix} \frac{2\sqrt{5}}{5\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} \\ 0 & \frac{-\sqrt{5}}{5\sqrt{6}} \end{pmatrix}$$

$$C'' = \frac{C'}{\|C'\|} = \begin{pmatrix} \frac{-\sqrt{6}}{15} & \frac{\sqrt{6}}{30} \\ \frac{-2\sqrt{6}}{5} & \frac{\sqrt{6}}{30} \end{pmatrix}, \quad D'' = \frac{D'}{\|D'\|} = \begin{pmatrix} \frac{-4}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

is an orthonormal basis.