HOMEWORK 2

(Math 258 A, B)

1. Prove **Pascal's Identity**. Namely, show that: (20 pts)

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}, \text{ where } k \le n .$$

(*Note*: Prove the identity by algebraic methods, <u>not</u> combinatorial methods).

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!k}{k(k-1)!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k)!(n-k+1)} \\
&= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!} \\
&= \frac{n!(k+n-k+1)}{k!(n-k+1)!} \\
&= \frac{n!(n+1)}{k!(n-k+1)!} \\
&= \frac{(n+1)!}{k!(n+1-k)!} \\
&= \binom{n+1}{k}
\end{aligned}$$

2. Prove the **Hexagon Identity**. Namely, show that: (20 pts)

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$

$$\begin{aligned}
\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} &= \left[\frac{(n-1)!}{(k-1)!(n-1-k+1)!}\right]\left[\frac{n!}{(k+1)!(n-k-1)!}\right]\left[\frac{(n+1)!}{k!(n+1-k)!}\right] \\
&= \frac{(n-1)!}{(k-1)!(n-1-k+1)!}\,\frac{n!}{(k+1)!(n-k-1)!}\,\frac{(n+1)!}{k!(n+1-k)!} \\
&= \frac{(n-1)!}{k!(n-1-k)!}\,\frac{n!}{(k-1)!(n-k+1)!}\,\frac{(n+1)!}{(k+1)!(n+1-k-1)!}
\end{aligned}$$

$$= \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$

3. Establish **Bolferoni's Inequality**. Namely, show that: $P(E \cap F) \geq P(E) + P(F) - 1$, where E and F events. (20 pts)

$P(E \cup F) = P(E) + P(F) - P(E \cap F)$
But also, $P(E \cup F) \leq 1$
i.e. $P(E) + P(F) - P(E \cap F) \leq 1$
$\Rightarrow P(E \cap F) \geq P(E) + P(F) - 1$

4. Two events E and F are called **independent** if and only if $P(E \cap F) = P(E)P(F)$. Show that if E and F are independent, then $\overline{E}$ and $\overline{F}$ are also independent. (20 pts)

$$\begin{aligned}
P(\overline{E} \cap \overline{F}) &= P(\overline{E \cup F}) \qquad \text{(by De Morgan's)} \\
&= 1 - P(E \cup F) \\
&= 1 - (P(E) + P(F) - P(E \cap F)) \\
&= 1 - P(E) - P(F) + P(E \cap F) \\
&= 1 - P(E) - P(F) + P(E)P(F) \\
&= 1 - P(F) - P(E) + P(E)P(F) \\
&= 1 - P(F) - P(E)(1 - P(F)) \\
&= (1 - P(F))(1 - P(E)) \\
&= P(\overline{E})P(\overline{F})
\end{aligned}$$

i.e. $\overline{E}$ and $\overline{F}$ are also independent

5. Determine whether the relation $R = \{(x, y) \in \Re \times \Re \mid xy \geq 0\}$ is: (20 pts)

(a) Reflexive: YES, because $(x, x) \in R, \forall x$. Indeed, $x \cdot x = x^2 \geq 0$.

(b) Symmetric: YES, because $(x, y) \in R \Rightarrow xy \geq 0 \Rightarrow yx \geq 0 \Rightarrow (y, x) \in R$.

(c) Antisymmetric: NO, because $(1,0) \in R$ and $(0,1) \in R$, but $1 \neq 0$.

(d) Transitive: NO, because $(1,0) \in R$ and $(0,-1) \in R$, but $(1,-1) \notin R$.