Test 3 (Key)

(Math 140)

1. **Find the eigenvalues and eigenvectors of** $A = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$**.**

$$\det\begin{vmatrix} -1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \;\Rightarrow\; (-1-\lambda)(2-\lambda)-1 = 0 \;\Rightarrow\; \lambda^2-\lambda-3 = 0$$

So, $\lambda = \dfrac{1\pm\sqrt{1-(4)(1)(-3)}}{2} = \dfrac{1\pm\sqrt{13}}{2}$ are the eigenvalues of A.

To find the eigenvectors we will solve for x and y the equation:

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{1\pm\sqrt{13}}{2}\right)\begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\begin{cases} -x+y = \dfrac{1}{2}x+\dfrac{\sqrt{13}}{2}x \\ x+2y = \dfrac{1}{2}y+\dfrac{\sqrt{13}}{2}y \end{cases} = \begin{cases} y = \left(\dfrac{3}{2}+\dfrac{\sqrt{13}}{2}\right)x \\ y = \left(\dfrac{3}{2}+\dfrac{\sqrt{13}}{2}\right)x \end{cases}$$

So, the first eigenvector is $\vec{u} = \begin{pmatrix} x \\ \left(\dfrac{3}{2}+\dfrac{\sqrt{13}}{2}\right)x \end{pmatrix}$.

Similarly, $\begin{cases} -x+y = \dfrac{1}{2}x-\dfrac{\sqrt{13}}{2}x \\ x+2y = \dfrac{1}{2}y-\dfrac{\sqrt{13}}{2}y \end{cases} = \begin{cases} y = \left(\dfrac{3}{2}-\dfrac{\sqrt{13}}{2}\right)x \\ y = \left(\dfrac{3}{2}-\dfrac{\sqrt{13}}{2}\right)x \end{cases}$

So, other eigenvector is $\vec{u} = \begin{pmatrix} x \\ \left(\dfrac{3}{2}-\dfrac{\sqrt{13}}{2}\right)x \end{pmatrix}$.

2. **Find the volume of the parallelepid S determined by the vectors** $\vec{u} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$**,**

$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ **and** $\vec{w} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$**. Also, find the area of the parallelogram determined by** $\vec{u}$ **and** $\vec{v}$ **.**

The volume of the parallelepid is given by: $V(S) = \left|(\vec{u} \times \vec{v}) \bullet \vec{w}\right|$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 2 & 0 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -(-2-0)j = 2j$$

$$(\vec{u} \times \vec{v}) \bullet \vec{w} = 0 \cdot 3 + 2 \cdot 2 + 0 \cdot 0 = 4$$

So, $V(S) = |4| = 4$.

The area of the base of the parallelepid is given by: $A(S) = \|\vec{u} \times \vec{v}\|$

So, $A(S) = \|\vec{u} \times \vec{v}\| = \|\begin{vmatrix} i & j & k \\ 2 & 0 & 0 \\ 1 & 0 & -1 \end{vmatrix}\| = \| -(-2-0)j \| = \sqrt{0+4+0} = 2$

3. **Find the parametric equations of the line *l* that passes through** $P = (1,1,3)$ **and is perpendicular to the plane *M* with equation** $2x + 3y - z = 2$ **.**

The normal to the plane is $N = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

The equation of the line is:
$$\begin{aligned} x &= P_1 + N_1 t \\ y &= P_2 + N_2 t \\ z &= P_3 + N_3 t \end{aligned}$$

So the line perpendicular to the plane M that passes through P is:
$$\begin{aligned} x &= 1 + 2t \\ y &= 1 + 3t \\ z &= 3 - t \end{aligned}$$

4. Identify the type of linear map the following represent.

(a) $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$

Scaling

(b) $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Reflection (about the *xy*-plane)

(c) $C = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Rotation (around the *z*-axis)

(d) $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Projection (on the *yz*-plane)

5. **Rotate the vector** $\vec{u} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ **around the vector** $\vec{a} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$ **by** $180°$**.**

(Note: Use this matrix: $A = \begin{pmatrix} a_1^2 + c(1-a_1^2) & a_1a_2(1-c) - a_3s & a_1a_3(1-c) + a_2s \\ a_1a_2(1-c) + a_3s & a_2^2 + c(1-a_2^2) & a_2a_3(1-c) - a_1s \\ a_1a_3(1-c) - a_2s & a_2a_3(1-c) + a_1s & a_3^2 + c(1-a_3^2) \end{pmatrix}$)

Since we are rotating by $180° \Rightarrow c = \cos(180°) = -1$ and $s = \sin(180°) = 0$.

So,

$$A = \begin{pmatrix} \frac{1}{2} - \frac{1}{2} & 1-0 & 0+0 \\ 1+0 & \frac{1}{2} - \frac{1}{2} & 0-0 \\ 0-0 & 0+0 & 0-1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Hence, $\vec{u}\,' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$