Test 4

(Math 140)

1. Given $A = \begin{pmatrix} -1 & 2 & 0 \\ 1 & -3 & 2 \\ 2 & -6 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, examine which of the two is invertible without calculating any inverses. (10 pts)

$$\det(A) = -1(-12+12) - 2(4-4) + 0(-6+6) = 0$$

Since, $|A| = 0 \Rightarrow A$ does not have an inverse.

$$\det(B) = 2(-1-0) - 1(0-0) + 0(0-0) = -2$$

Since, $|A| = -2 \Rightarrow A$ is invertible.

2. True or False: $|A+B| = |A| + |B|$. If your answer is False, make sure you provide a counterexample. (10 pts)
(*Note*: You can use 2x2 matrices for a counterexample)

Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Then, $A + B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which gives $|A+B| = 0$

But on the other hand, $|A| = 1$, $|B| = 1$

So, $|A+B| \neq |A| + |B|$

3. Consider the triangle with vertices $A = (1,1,0)$, $B = (0,0,0)$, and $C = (2,0,0)$. (30 pts)

(a) Find the 4×4 matrix that first rotates the triangle ABC by $180°$ about the y-axis, and then translates it by (0,2,0).

Let R be the matrix that rotates a vector by $180°$ around the y-axis. Then:

$$R = \begin{pmatrix} \cos 180 & 0 & \sin 180 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin 180 & 0 & \cos 180 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let T be the matrix that translates a vector by (0,2,0). Then:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So, the matrix that first rotates a vector by $180°$, and then translates it by (0,2,0) is:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) Use the above matrix to find the coordinates of the shifted triangle.

$$A' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

$$B' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$C' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

(c) Make all relevant graphs that show the original triangle, the rotation, and the shifted triangle.

Clear

4. Given the basis $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, examine whether it is an orthonormal basis. If your answer is **no**, then use the *Gram-Schmidt Orthogonalization Process* to construct an orthonormal basis from the basis above. (30 pts)

(*Note*: Take the inner product to be the usual **dot** product)

The basis is not orthonormal, because $\vec{u} \cdot \vec{w} = 1 + 0 + 0 = 1 \neq 0$.

Let, $\vec{u}' = \vec{u}$

$$\vec{v}' = \vec{v} - \frac{\langle \vec{v}, \vec{u}' \rangle}{\|\vec{u}'\|^2} \vec{u}' = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} - \frac{0}{1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\vec{w}' = \vec{w} - \frac{\langle \vec{w}, \vec{v}' \rangle}{\|\vec{v}'\|^2} \vec{v}' - \frac{\langle \vec{w}, \vec{u}' \rangle}{\|\vec{u}'\|^2} \vec{u}' = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{0}{1} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Then, $\vec{u}', \vec{v}', \vec{w}'$ is an orthogonal basis.

Furthermore, $\vec{u}'' = \frac{\vec{u}'}{\|\vec{u}'\|} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}'' = \frac{\vec{v}'}{\|\vec{v}'\|} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$, $\vec{w}'' = \frac{\vec{w}'}{\|\vec{w}'\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ which is an orthonormal basis.

5. (a) True or False:

   (i) A Bezier curve is linear if and only if all control points lie on the curve.

   True

   (ii) Bezier curves are preserved under affine transformations.

   True

   (b) Find the point on the Bezier curve determined by A = (2,2), B = (4,2), and C = (5,0) that corresponds to t = 1/3. (20 pts)

Let, $L(u,v,t) = (1-t)u + tv$

The point on the Bezier curve for $t = a$ is: $L(L(A,B,a), L(B,C,t), a)$.

$$L(A,B,1/3) = (1-1/3)A + 1/3B = \left(\frac{2}{3}\right)(2,2) + \left(\frac{1}{3}\right)(4,2) = \left(\frac{8}{3} \quad 2\right)$$

$$L(A,B,1/3) = (1-1/3)B + 1/3C = \left(\frac{2}{3}\right)(4,2) + \left(\frac{1}{3}\right)(5,0) = \left(\frac{13}{3} \quad \frac{4}{3}\right)$$

Then, $L(L(A,B,1/3), L(B,C,t), 1/3) = \left(\frac{2}{3}\right)\left(\frac{8}{3}, 2\right) + \left(\frac{1}{3}\right)\left(\frac{13}{3}, \frac{4}{3}\right) = \left(\frac{29}{9}, \frac{16}{9}\right).$