Test 4

(Math 258, A)

1. Show that $k\binom{n}{k} = n\binom{n-1}{k-1}$, where $n, k \in N - \{0\}$ (10 pts)

$$n\binom{n-1}{k-1} = n\left(\frac{(n-1)!}{(k-1)!(n-1-(k-1))!}\right)$$
$$= n\left[\frac{(n-1)!}{(k-1)!(n-k)!}\right]$$
$$= \frac{n(n-1)!}{(k-1)!(n-k)!}$$
$$= \frac{n!}{(k-1)!(n-k)!}$$
$$= \frac{kn!}{k(k-1)!(n-k)!}$$
$$= k\left[\frac{n!}{k!(n-k)!}\right]$$
$$= k\binom{n}{k}$$

2. What is the coefficient of x^5y^6 in $(2x + y)^{11}$, without doing expansion? (10 pts)

$$\binom{11}{6}2^5 = \frac{11!}{6!(11-6)!} \cdot 2^5 = 462 \cdot 32$$

So the coefficient is $462 \cdot 32 = 14{,}784$.

3. What is the probability that the sum of the numbers on two dice is even when they are rolled? (10 pts)

Whatever the number of spots showing on the first die, the sum will be even if and only if the number of spots showing on the second has the same parity (even or odd) as the first. Since there are 3 even faces (2, 4, 6) and 3 odd faces (1, 3, 5), the probability is 3/6 = ½.

4. Find the probability of each outcome when a loaded die is rolled, if a 3 is twice as likely to appear as each of the other five numbers on the die. (10 pts)

   $P_3 = 2P_i$, $i = 1, 2, 4, 5, 6$, i.e. $P_3 = 2P_1$, $P_3 = 2P_2$, $P_3 = 2P_4$, $P_3 = 2P_5$, $P_3 = 2P_6$.

   Also, $P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 1$

   This gives us the 5x5 system of equations:

   $3P_1 + P_2 + P_4 + P_5 + P_6 = 1$
   $P_1 + 3P_2 + P_4 + P_5 + P_6 = 1$
   $P_1 + P_2 + 3P_4 + P_5 + P_6 = 1$
   $P_1 + P_2 + P_4 + 3P_5 + P_6 = 1$
   $P_1 + P_2 + P_4 + P_5 +3 P_6 = 1$

   Using the first 2 equations, we get that:

   $P_1 – P_2 = 0$, ie $P_1 = P_2$
   $P_2 – P_4 = 0$, ie $P_2 = P_4$
   $P_4 – P_5 = 0$, ie $P_4 = P_5$
   $P_5 – P_6 = 0$, ie $P_5 = P_6$

   This means that , ie $P_1 = P_2 = P_4 = P_5 = P_6$.

   Putting this back into the first equation, we get that

   $3P_1 + 4P_1 = 1$ which implies that $P_1 = 1/7$.
   So $P_1 = P_2 = P_4 = P_5 = P_6 = 1/7$.

   And since $P_3 = 2P_1$, we get that $P_3 = 2/7$.

5. Given a family with two children, what is the probability that both children are girls if the older child is a girl? (10 pts)

   If a family only has 2 children, then there are 4 equally likely outcomes. BB, BG, GB, and GG. There are 2 ways to have an oldest girl, GB and GG. So the odds of having two girls is 2/4 = ½.

6. Determine whether the relation $R = \{(x, y) \in \Re \times \Re \mid x + y = 0\}$ is: (20 pts)
   a. Reflexive

      No, because $(1,1) \notin R$.

   b. Symmetric

Yes, since addition is commutative. i.e. if $x+y=0 \Rightarrow y+x=0$.

c. Antisymmetric

It is not antisymmetric because $(1,-1) \in R$ and $(-1,1) \in R$, but $1 \neq -1$.

d. Transitive

It is not transitive because $(1,-1) \in R$ and $(-1,1) \in R$, but $(1,1) \notin R$.

7. Given the relations $R_1 = \{(x,y) \in \Re \times \Re \mid x \leq y\}$, $R_2 = \{(x,y) \in \Re \times \Re \mid x < y\}$ and $R_3 = \{(x,y) \in \Re \times \Re \mid x \neq y\}$, find the following: (20 pts)

a. $R_1 \cup R_2$

For $(x,y) \in R_1 \cup R_2$, we must have $x \leq y$ or $x < y$, i.e. $x \leq y$ which is R_1.

b. $R_1 \cap R_3$

For (x,y) to be in $R_1 \cap R_3$, we must have $x \leq y$ and $x \neq y$. Since this happens precisely when $x < y$, we see that the answer is R_2.

c. $R_2 - R_3$

Recall that $R_2 - R_3 = R_2 \cap \overline{R_3}$. But $\overline{R_3} = \{(x,y) \in \Re \times \Re \mid x = y\}$. But it is impossible for $x < y$ and $x = y$ to hold at the same time. So the answer is $\varnothing$, i.e. the relation never holds.

d. $R_1 \circ R_3$

For (x,z) to be in $R_1 \circ R_3$, we must find an element y such that $(x,y) \in R_1$ and $(y,z) \in R_3$. This means that $x \leq y$ and $y \neq z$. Clearly this is true $\forall (x,z)$ by simply choosing x to be small enough. Therefore we have that $R_1 \circ R_3 = \Re^2$, the relation always holds.

8. A Relation R on a set A is called irreflexive if and only if $\forall a \in A, (a,a) \notin A$. Can a relation on a set be neither reflexive nor irreflexive? (10 pts)
(*Note*: Whether your answer is yes or no, provide an example.)

Yes, for $R = \{(1,1), (1,2)\}$ on $A = \{1,2\}$. Since $(1,1) \in R$, R is not irreflexive. Since $(2,2) \notin R$, R is not reflexive. Hence, R is neither irreflexive or reflextive.