Test 4

(Math 258, B)

1. Show that $\binom{n+1}{k} = (n+1)\dfrac{\binom{n}{k-1}}{k}$, where $n,k \in \mathbf{N} - \{\mathbf{0}\}$

   Start with the LHS:
   $$\binom{n+1}{k} = \frac{(n+1)!}{k!(n+1-k)!}$$

   Start with the RHS:
   $$(n+1)\frac{\binom{n}{k-1}}{k} = (n+1)\frac{\frac{n!}{(k-1)!(n-k+1)!}}{k} = \frac{k(n+1)!}{(k-1)!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!}$$

   Since, RHS = LHS, both sides are equal.

2. What is the coefficient of x^7y^6 in $(x-2y)^{13}$, without doing the expansion?

   $$\binom{13}{6}(1)^7(-2)^6 = \binom{13}{6}(2)^6 = \frac{13!}{6!(7!)}(2)^6 = 109{,}824$$

3. If a fair coin is tossed 6 times, what is the probability of getting exactly 4 heads?

   $$P = \frac{\binom{6}{4}}{2^6} = \frac{\frac{6!}{4!(2!)}}{64} = \frac{15}{64}$$

4. Find the probability of each outcome when a loaded die is rolled, if a 3 is twice as likely to appear as each of the other five numbers on the die.

   We have that : $P_3 = 2P_1, P_3 = 2P_2, P_3 = 2P_4, P_3 = 2P_5, P_3 = 2P_6$
   and
   $P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 1$

This gives us the 5x5 system of equations:

$3P_1 + P_2 + P_4 + P_5 + P_6 = 1$
$P_1 + 3P_2 + P_4 + P_5 + P_6 = 1$
$P_1 + P_2 + 3P_4 + P_5 + P_6 = 1$
$P_1 + P_2 + P_4 + 3P_5 + P_6 = 1$
$P_1 + P_2 + P_4 + P_5 +3 P_6 = 1$

Using the first 2 equations, we get that:

$P_1 – P_2 = 0$, i.e. $P_1 = P_2$
$P_2 – P_4 = 0$, i.e. $P_2 = P_4$
$P_4 – P_5 = 0$, i.e. $P_4 = P_5$
$P_5 – P_6 = 0$, i.e. $P_5 = P_6$

This means that $P_1 = P_2 = P_4 = P_5 = P_6$.

Putting this back into the first equation, we get that:

$3P_1 + 4P_1 = 1$ which implies that $P_1 = 1/7$.
So, $P_1 = P_2 = P_4 = P_5 = P_6 = 1/7$.

And since $P_3 = 2P_1$, we get that $P_3 = 2/7$.
Hence, $P_3 = \frac{2}{7}$ and $P_1 = P_2 = P_4 = P_5 = P_6 = \frac{1}{7}$

5. Two dice are rolled. Find the probability that the sum is 10 or greater.

   The following 6 pairs add up to 10 or greater:
   $(4,6), (6,4), (5,5), (6,5), (5,6), (6,6)$
   Hence, the probability = $\frac{6}{36} = \frac{1}{6}$

6. Determine whether the relation $R = \{(x, y) \in \Re \times \Re \mid x = 2y\}$ is:

   a. Reflexive: No, $(1,1) \notin R$
   b. Symmetric: No, $(2,1) \in R$ but $(1,2) \notin R$
   c. Antisymmetric: Yes, $(x, y) \in R \to x = 2y$ and $(y, x) \in R \to y = 2x$
   $\Rightarrow x = 4x$, hence $x = 0 \Rightarrow y = 0$, i.e. $x = y$
   d. Transitive: No, $(2,1) \in R$ and $(1,½) \in R$, but $(2,½) \notin R$

7. Given the relations $R_1 = \{(x,y) \in \Re \times \Re \mid x \le y\}$, $R_2 = \{(x,y) \in \Re \times \Re \mid x < y\}$ and $R_3 = \{(x,y) \in \Re \times \Re \mid x \ne y\}$, find the following:
   a. $R_1 \cup R_2 = R_1$
   b. $R_1 \cap R_3 = R_2$
   c. $R_2 - R_3 = \varnothing$
   d. $R_1 \circ R_3 = \{(x,z) \text{ st } (x,y) \in R_1 \wedge (y,z) \in R_3\}$ (where $x \le y \wedge y \ne z$)
      $= \Re^2$

8. A relation R on a set A is called irreflexive if and only if $\forall a \in A, (a,a) \notin R$. Can a relation on a set be neither reflexive nor irreflexive?
   (Note: Whether your answer is **yes** or **no**, make sure you provide an example)

   Yes, let $A = \{1,2\}, R = \{(1,1),(1,2)\}$
   Since, $(1,1) \in R$, but $(2,2) \notin R$, R is neither irreflexive nor reflexive.