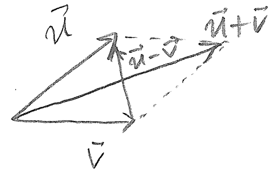
Homework 1 Key

(1)



$\vec{u}+\vec{v} \perp \vec{u}-\vec{v}$ iff $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = 0$.

$$\begin{aligned}
(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) &= (u_1+v_1, u_2+v_2)\cdot(u_1-v_1, u_2-v_2) \\
&= (u_1+v_1)(u_1-v_1) + (u_2+v_2)(u_2-v_2) \\
&= u_1^2 - v_1^2 + u_2^2 - v_2^2 \\
&= u_1^2 + u_2^2 - (v_1^2+v_2^2) \\
&= \|\vec{u}\|^2 - \|\vec{v}\|^2 \\
&\overset{\|\vec{u}\|=\|\vec{v}\|}{=} \|\vec{u}\|^2 - \|\vec{u}\|^2 \\
&= 0
\end{aligned}$$

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(2)

(a) $1+2 = (1)(2) = 2$

(b) $0 \cdot 2 = 2^0 = 1$

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(3)

$\dim \mathbb{C} = 2$. That's because $\mathbb{C}$ has a basis that consists of 2 vectors, namely $\{1, i\}$.
(We can easily show that $1, i$ span $\mathbb{C}$ and that they are lin. indep.)

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(4) To show that p_1, p_2, p_3 span $P_2[x]$ and that they are lin. indep.

<u>span</u>: Let $\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 = q$, $q \in P_2[x]$

$$\Rightarrow \lambda_1(1+x) + \lambda_2(3) + \lambda_3(x+x^2) = ax^2 + bx + c$$

$$\Rightarrow \lambda_3 x^2 + (\lambda_1 + \lambda_3)x + \lambda_1 + 3\lambda_2 = ax^2 + bx + c$$

$$\Rightarrow \left.\begin{array}{l} \lambda_3 = a \\ \lambda_1 + \lambda_3 = b \\ \lambda_1 + 3\lambda_2 = c \end{array}\right\} \Rightarrow \lambda_3 = a,\ \lambda_1 = b-a,\ \lambda_2 = \frac{c-b+a}{3}$$

ie, $q(x) = (b-a)(1+x) + \frac{c-b+a}{3}(3) + a(x+x^2)$

ie p_1, p_2, p_3 span $P_2[x]$.

<u>lin. indep</u>: Let $\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 = 0$

$$\Rightarrow \lambda_1(1+x) + \lambda_2(3) + \lambda_3(x+x^2) = 0x^2 + 0x + 0$$

$$\Rightarrow \lambda_3 x^2 + (\lambda_1 + \lambda_3)x + \lambda_1 + 3\lambda_2 = 0x^2 + 0x + 0$$

$$\Rightarrow \left.\begin{array}{l} \lambda_3 = 0 \\ \lambda_1 + \lambda_3 = 0 \\ \lambda_1 + 3\lambda_2 = 0 \end{array}\right\} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0.$$

ie, p_1, p_2, p_3 lin. indep.

Since p_1, p_2, p_3 span $P_2[x]$ and are lin. indep. they are a basis for $P_2[x]$.

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(5) (a) $A^2 = AA = \begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix}\begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

So, $A^n = 0$, $\forall n \geq 2$.

(b) $$\left.\begin{aligned} AB &= \begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 12 \\ 2 & -8 \end{pmatrix} \\ BA &= \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ -6 & -9 \end{pmatrix} \end{aligned}\right\} \neq$$

(c) $$\begin{aligned} B^2 - A - 2I &= \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 \\ -1 & -2 \end{pmatrix} - \begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -8 & -7 \\ 3 & 2 \end{pmatrix} \end{aligned}$$

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(6) (a) $(A + A^t)^t = A^t + (A^t)^t = A^t + A = A + A^t$

ie, $A + A^t$ symmetric.

(b) $(A - A^t)^t = A^t - (A^t)^t = A^t - A = -(A - A^t)$

ie, $A - A^t$ antisymmetric.

(c) $A = \dfrac{A + A^t}{2} + \dfrac{A - A^t}{2}$

But from (b) and (c), $A + A^t$ and $A - A^t$ are symmetric and antisymmetric respectively.

So, $\dfrac{A+A^t}{2}$ and $\dfrac{A-A^t}{2}$ are symm. and antisymm. respectively.

Hence, A is the sum of a symm. and an antisymm. matrix.

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