HOMEWORK 1
(Math 250 A, B)

1. Let W_1 and W_2 be subspaces of a vector space *V*. Then: (20 pts)
   (a) Show that $W_1 \cap W_2$ is a subspace of *V*.
   (b) Is $W_1 \cup W_2$ a subspace of V? If your answer is **no**, provide a counterexample.

2. Show that the set of all symmetric matrices, i.e.É $W = \{A \in M_{2\times 2}(\mathrm{R}) \mid A^t = A\}$, is a subspace of $M_{2\times 2}(\mathrm{R})$. Exhibit a basis for *W* and find the *dim*(*W*) . (20 pts)
   (*Hint*: Try to find the form that a symmetric matrix *A* has in general. That will help you find the elements that *A* can be decomposed into, in other words the elements that span *W*)

3. Let $W = \{p(x) \in P_2[x] \mid p(x) = p(-x)\}$. Show that *W* is a subspace of $P_2[x]$. Exhibit a basis for *W* and find the *dim*(*W*) . (20 pts)
   (*Hint*: Try to find the form that a polynomial *p*(*x*) in *W* has in general. That will help you find the elements that *p*(*x*) can be decomposed into, in other words the elements that span *W*)

4. Let *S* be a set. For each $s \in S$ the function $\chi_s : S \to \mathrm{R}$ defined by: (20 pts)
$$\chi_s(t) = \begin{cases} 1, & t = s \\ 0, & t \neq s \end{cases} \text{É}$$
   is called the **characteristic function** of s. Denote the set of all characteristic functions by *X*. Consider also the set of all functions $f : S \to \mathrm{R}$, denoted by $F(S)$.
   Show the following:
   (a) If *S* is finite, say $S = \{s_1, s_2, ..., s_n\}$, then $\chi_{s_1}, \chi_{s_2}, ..., \chi_{s_n}$ form a basis for $F(S)$.
   (b) If *S* is infinite, then the *dim*($F(S)$) $= \infty$. Could $\chi_{s_1}, \chi_{s_2}, ..., \chi_{s_n}, ...$ serve as a basis for $F(S)$?
   (*Hint*: For the 1st-half of (b), use a proof by contradiction, considering a maximal independent set and also the fact that $\chi_{s_1}, \chi_{s_2}, ..., \chi_{s_n}, ...$ could be shown to be linearly independent. For the 2nd-half of (b), find a function that cannot be expressed as finite comb's of $\chi_{s_1}, \chi_{s_2}, ..., \chi_{s_n}, ...$)

5. If $f_1(x), f_2(x), ..., f_n(x) \in C^{(n-1)}(\mathrm{R})$, i.e. they are ($n-1$)-times differentiable functions, then the determinant defined by: (20 pts)
$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$
   is called the **Wronskian** of $f_1(x), f_2(x), ..., f_n(x)$. This determinant is useful for showing us whether the functions $f_1(x), f_2(x), ..., f_n(x)$ are linearly independent

or not. For example, if $W(x) \neq 0$, then $f_1(x), f_2(x), \ldots, f_n(x)$ are linearly independent.

Show the following:

(a) The functions 1, e^x, e^{2x} are linearly independent functions in $C^2(\mathrm{R})$.

(b) Is it true that if $W(x) = 0$, then $f_1(x), f_2(x), \ldots, f_n(x)$ are linearly dependent? If your answer is **no**, provide a counterexample.

(*Hint*: For (b), try $f_1(x) = x^3$, $f_2(x) = |x^3|$É)